

HOW TO DETERMINE A K3 SURFACE FROM A FINITE AUTOMORPHISM

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ABSTRACT. In this article we pursue the question when an automorphism determines a (complex) K3 surface up to isomorphism. We prove that if the automorphism is finite non-symplectic and the transcendental lattice small, then the isomorphism class of the K3 surface is determined by an n -th root of unity and an ideal in $\mathbb{Z}[\zeta_n]$. As application we give a generalization of Vorontsov's theorem and the classification of purely non-symplectic automorphisms of high order. Furthermore, we prove that there exist infinitely many K3 surfaces with a symplectic and a non-symplectic automorphism of order 5. If they commute such a K3 surface is unique. We give a description of its Néron-Severi lattice.

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1. INTRODUCTION

An algebraic K3 surface X is a smooth projective surface over an algebraically closed field $k = \overline{k}$ with vanishing irregularity $q = h^1(X, \mathcal{O}_X)$ and trivial canonical bundle, i.e. $\omega_X = \mathcal{O}_X$. A complex (holomorphic) K3 surface is a smooth, compact, complex surface with vanishing irregularity and trivial canonical bundle. It is not necessarily algebraic. The results of this article concern mostly algebraic K3 surfaces.

An automorphism f of a K3 surface is called symplectic if it acts trivially on the global holomorphic 2-forms, $f^*|H^0(X, \Omega_X^2) = \text{id}$, and non-symplectic otherwise. Furthermore we call f purely non-symplectic if all non-trivial powers are non-symplectic. Note that K3 surfaces admitting a non-symplectic automorphism of finite order are always algebraic [36, 3.1]. Being symplectic or not governs the deformation behavior of the automorphism. One expects that a symplectic automorphism deforms (at least) in $\text{rk } T(X) - 2$ dimensions, while a non-symplectic automorphism acting by order n on the holomorphic 2-form deforms in $\text{rk } T(X) / \varphi(n) - 1$ dimensions where φ is the Euler totient function. In order to determine a K3 surface by some fixed data d , the pair (X, d) should not deform. In the symplectic case this means that $\text{rk } T(X) = 2$. There one can reconstruct the K3 surface up to isomorphism from the (oriented) transcendental lattice by means of a Shioda-Inose structure [46]. This article is concerned with the non-symplectic case, i.e. $\text{rk } T(X) = \varphi(n)$.

We prove that if X is a complex K3 surface and $f \in \text{Aut}(X)$ of finite order with

$$\text{rk } T(X) = \varphi(\text{order}(f|H^{2,0}(X))),$$

then the action of f on $H^0(X, \Omega_X^2)$ and $\text{NS}^\vee(X)/\text{NS}(X)$ determine the isomorphism class of X . This is encoded in a root of unity and an ideal in $\mathbb{Z}[\zeta_n]$. If f is of infinite order, we need the additional data of a primitive embedding $T(X) \hookrightarrow L_{K3}$. In many cases it is unique.

We shall give two applications of this theorem throughout this note. The first is a generalization of Vorontsov's Theorem 6.1 and related results. The second application focuses on K3 surfaces admitting a symplectic and a non-symplectic automorphism of order 5.

Symplectic automorphisms of finite order on complex K3 surfaces were studied first in [36], while non-symplectic automorphisms of prime order and their fixed points were classified in [4]. The authors proved that the moduli space \mathcal{M}_{K3}^5 of K3 surfaces admitting a non-symplectic automorphism of order 5 has two irreducible components distinguished by whether the automorphism fixes a curve pointwise or not. In [19] A. Garbagnati and A. Sarti showed that the moduli space of complex K3 surfaces admitting both a symplectic and a non-symplectic automorphism of order 5 is zero dimensional. So at most countably many such surfaces may exist. The authors then gave a single example lying in the intersection of the two irreducible components of \mathcal{M}_{K3}^5 . It is given as the minimal resolution S of the double cover of \mathbb{P}^2 branched over the sextic $x_0(x_0^5 + x_1^5 + x_2^5)$. The (non)-symplectic automorphisms are induced by multiplying coordinates with 5th roots of unity. In particular the

automorphisms commute. One can ask if this example is unique. We give two different answers to this question:

- No, there exist infinitely many non-isomorphic complex K3 surfaces with both a symplectic and non-symplectic automorphism of order 5.
- Yes, if the two automorphisms commute.

The proofs are carried out by reformulating all statements in terms of Hodge structures and lattices.

2. COMPLEX K3 SURFACES

Let X be a complex K3 surface. Its second singular cohomology equipped with the cup product is an even unimodular lattice

$$H^2(X, \mathbb{Z}) \cong 3U \oplus 2E_8 =: L_{K3}$$

of signature $(3, 19)$. Such a lattice is unique up to isometry. Definitions and properties of lattices are given in the next section. By the Hodge decomposition

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

where $H^{i,j}(X) \cong H^j(X, \Omega_X^i)$, $H^{i,j}(X) = \overline{H^{j,i}(X)}$ and $H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^\perp$ has signature $(1, 19)$.

Conversely, the Hodge structure of a K3 surface determines it up to isomorphism as is reflected by the Torelli theorems.

Theorem 2.1. *[7, VIII 11.1] Let X, Y be complex K3 surfaces and*

$$f : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$$

an isometry of lattices whose \mathbb{C} -linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$. Then $X \cong Y$. If moreover f maps effective classes on X to effective classes on Y , then $f = F^$ for a unique isomorphism $F : Y \rightarrow X$.*

The Hodge decomposition is determined by the period $H^{2,0}(X) =: \omega$ and the space of all such Hodge structures on L_{K3} ,

$$\Omega(L_{K3}) = \{[\omega] \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\},$$

is called the period domain.

A marked K3 surface is a complex K3 surface X together with an isometry $\phi : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$ called marking. Two marked K3 surfaces (X, ϕ_X) and (Y, ϕ_Y) are equivalent if there is an isomorphism $f : X \rightarrow Y$ with $\phi_Y = \phi_X \circ f^*$. Let \mathcal{M} be the moduli space of equivalence classes of marked K3 surfaces.

Theorem 2.2. *[7, VIII 12.2] The period map*

$$\pi : \mathcal{M} \rightarrow \Omega(L_{K3}), \quad (X, \phi) \mapsto \phi(H^{2,0}(X))$$

is surjective.

By Lefschetz' Theorem on $(1, 1)$ classes we can recover the Néron-Severi group from the period as

$$\text{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

Its rank ρ is called the Picard number of X . The transcendental lattice is defined as the smallest primitive sublattice $T \subseteq H^2(X, \mathbb{Z})$ whose complexification contains

$H^{2,0} \subseteq T \otimes \mathbb{C}$. The surface X is projective if and only if NS has signature $(1, \rho - 1)$. In this case $T(X) = \text{NS}(X)^\perp$. If we consider just a single K3 surface X , we will usually omit the (X) from notation and just write $H^{i,j}$, NS, T , etc. From now on all K3 surfaces are assumed to be projective.

3. LATTICES

A *lattice* is a finitely generated free abelian group $L \cong \mathbb{Z}^n$ together with a non-degenerate bilinear pairing

$$L \times L \mapsto \mathbb{Z}, \quad (x, y) \mapsto x \cdot y$$

It is called even if $x^2 := x \cdot x \in 2\mathbb{Z}$ for all $x \in L$. The pairing induces an isomorphism

$$\text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee := \{x \in L \otimes \mathbb{Q} \mid x \cdot L \subseteq \mathbb{Z}\}$$

with the dual lattice L^\vee . The quotient $L^\vee/L =: D(L)$ (or D_L) is finite, abelian and called discriminant group. If D is a finite abelian group, the minimum number of generators of D is called the length $l(D)$ of D . Note that $l(D_L) \leq \text{rk } L$. If $D_L = 0$, we call the lattice L unimodular, and if $pD_L = 0$, we call L p -elementary. The discriminant group is equipped with a fractional form

$$b : D_L \times D_L \rightarrow \mathbb{Q}/\mathbb{Z}, \quad (\bar{x}, \bar{y}) \mapsto x \cdot y + \mathbb{Z}.$$

On an even lattice there is the discriminant form q given by

$$q : D_L \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad \bar{x} \mapsto x^2 + 2\mathbb{Z}.$$

The discriminant group decomposes as an orthogonal direct sum of p -groups

$$D(L) \cong \bigoplus_p D(L)_p.$$

By polarization $b|D(L)_p^2$ and $q|D(L)_p$ carry the same information for odd primes $p \neq 2$. If $D(L)_p$ is an \mathbb{F}_p -vector space, then $q|D(L)_p$ takes values in $\frac{2}{p}\mathbb{Z}/2\mathbb{Z}$. In this case, if $p \neq 2$, we denote by q_p the quadratic form $q_p(x) \equiv px^2 \pmod{p}$ on $D(L)_p$ with values in \mathbb{F}_p . Such a form is determined up to isometry by its determinant $\det(q_p) \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$ (cf. [20]). We call a quadratic/bilinear form module isotropic if the form vanishes identically.

We say that two lattices M and N are in the same *genus* if $N \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are isometric over the p -adic integers for all primes p and $N \otimes_{\mathbb{Z}} \mathbb{R} \cong M \otimes_{\mathbb{Z}} \mathbb{R}$ over the real numbers.

Theorem 3.1. [37, 1.9.4] *The signature (n_+, n_-) and discriminant form q determine the genus of an even lattice and vice versa.*

Locally this means that $N \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $q|D(L)_p$ carry the same information.

The local data at different primes is connected by the oddity formula

$$\text{sig}(L) + \sum_{p \geq 3} p\text{-excess}(L) \equiv \text{oddity}(L) \pmod{8}$$

where the $\text{sig}(L)$ is defined as $n_+ - n_-$. The p -excess is an invariant of $q|D(L)_p$ for $p \geq 3$ while the oddity is an invariant of the even part $q|D(L)_2$.

We need only the following simple cases:

- If L is an even lattice and $2 \nmid \det L$, then the oddity (L) is zero.
- If $D(L)_p$ is an \mathbb{F}_p -vector space, $p \neq 2$, then

$$p\text{-excess}(L) \equiv \dim D(L)_p(p-1) + 4k_p \pmod{8},$$

where $k_p = 1$ if the Legendre symbol $\left(\frac{\det q_p}{p}\right) = -1$, and zero else.

For precise definitions and a more detailed discussion of the classification of quadratic forms we refer to [14, Chapter 15].

3.1. Gluing isometries. An embedding of lattices $i : M \hookrightarrow L$ is called primitive if the cokernel is free. We call two primitive embeddings $i, j : M \hookrightarrow L$ isomorphic if there is a commutative diagram with $f \in O(L)$.

$$\begin{array}{ccc} M & \xhookrightarrow{i} & L \\ & \searrow j & \downarrow f \\ & & L \end{array}$$

We say that S embeds uniquely into L if all primitive embeddings are isomorphic. A (weakened) criterion for this to happen is given in the next theorem.

Theorem 3.2. [37, Theorem 1.14.4] [34, 2.8] *Let M be an even lattice of signature (m_+, m_-) and L an even unimodular lattice of signature (l_+, l_-) . If $l(D_M) + 2 \leq \text{rk } L - \text{rk } M$ and $l_+ > m_+, l_- > m_-$, then there is a unique primitive embedding of M into L .*

We mention the related

Theorem 3.3. [37, Theorem 1.14.2] *Let M be an even, indefinite lattice such that $\text{rk } M \geq 2 + l(D_M)$, then the genus of M contains only one class, and the homomorphism $O(M) \rightarrow O(q_M)$ is surjective.*

A different perspective on primitive embeddings is that of primitive extensions. Let $i : M \hookrightarrow L$ be a primitive embedding. Then $N := M^\perp$ is also a primitive sublattice and we call

$$M \oplus N \hookrightarrow L$$

a primitive extension. In particular L is an overlattice of $M \oplus N$. The bilinear form provides us with a chain of embeddings

$$M \oplus N \hookrightarrow L \hookrightarrow L^\vee \hookrightarrow M^\vee \oplus N^\vee$$

and

$$L/(M \oplus N) \hookrightarrow L^\vee/(M \oplus N) \hookrightarrow M^\vee/M \oplus N^\vee/N.$$

Since L is integral (even), $G = L/(M \oplus N)$ is an isotropic subspace with respect to $b_M \oplus b_N$ (respectively $q_M \oplus q_N$). Primitivity of the embeddings translates to the fact that G is a graph of a “glue map” ϕ defined on subgroups

$$(1) \quad D(M) \supseteq p_M(L)/M =: G_M \xrightarrow[\phi]{\cong} G_N := p_N(L)/N \subseteq D(N)$$

where p_N, p_M are the projections to N^\vee and M^\vee . Since G is isotropic, ϕ satisfies

$$q_M(x) = -q_N(\phi(x)) \pmod{2\mathbb{Z}} \quad \forall x \in G_M$$

in the even case or

$$b_M(x, y) = -b_N(\phi(x), \phi(y)) \pmod{\mathbb{Z}} \quad \forall x, y \in G_M$$

for odd lattices. Conversely the graph

$$G := \{x + \phi(x) \mid x \in G_M\}$$

of a glue map ϕ in $D(M) \oplus D(N)$ is an isotropic subspace and thus determines an overlattice L of $M \oplus N$. Since G is a graph of an isomorphism, $G \cap D(N) = 0$ and the embedding is primitive. In this way primitive extensions correspond bijectively to glue maps defined on subgroups. Since $G^\perp = L^\vee / (M \oplus N)$, we can compute the discriminant form of the overlattice via $G^\perp / G \cong L^\vee / L$. In this situation we write $L = M \oplus_\phi N$ and call G the glue of the primitive extension $M \oplus N \hookrightarrow L$.

Lemma 3.4.

$$|D_N / G_N| \cdot |D_M / G_M| = \det L$$

Proof. Divide the standard formula

$$\det M \det N = [L : M \oplus N]^2 \det L$$

by $[L : M \oplus N]^2$ and use the isomorphisms (1). \square

If L is unimodular, this recovers the well known fact that

$$D_M = G_M \xrightarrow[\phi]{\cong} G_N = D_N.$$

For example, $D_{T(X)} \cong D_{\text{NS}(X)}$ for a K3 surface X over \mathbb{C} .

Two isometries $f_M \in O(M)$, $f_N \in O(N)$ glue to an isometry $f_M \oplus_\phi f_N$ on L iff the induced action $\overline{f}_M \oplus \overline{f}_N$ in $D(M) \oplus D(N)$ preserves G , the graph of ϕ . Or equivalently if $\overline{f}_M, \overline{f}_N$ preserve G_M , respectively G_N , and $\phi \circ \overline{f}_M = \overline{f}_N \circ \phi$.

Example 3.5. We shall compute a simple example which we will meet again later. Consider two rank one lattices A and B generated by $a \in A, b \in B$ with $a^2 = -2$, $b^2 = -18$. Then A^\vee is generated by $a/2$ and B^\vee is generated by $b/18$. The 2-torsion part of their discriminant groups is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by $\overline{a/2} = a/2 + \mathbb{Z}a$ and $\overline{b/2} = b/2 + \mathbb{Z}b$. Up to sign there is a unique isomorphism $\overline{a/2} \mapsto \overline{b/2}$. Note that it is actually a glue map since

$$q_A(\overline{a/2}) + q_B(\overline{b/2}) = 1/2 + 2\mathbb{Z} - 9/2 + 2\mathbb{Z} = 2\mathbb{Z}.$$

Hence, its graph $G = \{\overline{0}, \overline{a/2} + \overline{b/2}\} \subseteq D(A) \oplus D(B)$ is isotropic and defines an (even) overlattice of $A \oplus B$ generated by $\{a, b, a/2 + b/2\}$. In the basis $(a, a/2 + b/2)$ its Gram matrix is $\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$.

Example 3.6. If X is a complex K3 surface, then

$$\text{NS} \oplus T \hookrightarrow H^2(X, \mathbb{Z})$$

is a primitive extension. Let f be an automorphism of X . It acts (by pullback) on the objects, $\text{NS}, T, H^2(X, \mathbb{Z}), D_{\text{NS}} = G_{\text{NS}} \cong G \cong G_T = D_T$ in a compatible way. If confusion is unlikely, we will denote all these actions by f .

Theorem 3.7. [29] *Let $f_i \in O(L_i)$ $i = 1, 2$ be a pair of lattice isometries, and let p be a prime number. Suppose*

- *Each discriminant group $D(L_i)_p$ is a vector space over \mathbb{F}_p ;*
- *The maps \bar{f}_i on $D(L_i)_p$ have the same characteristic polynomial $S(x)$ and*
- *$S(x) \in \mathbb{F}_p[x]$ is a separable polynomial, with $S(1)S(-1) \neq 0$.*

Then there is a gluing map $\phi_p : D(L_1)_p \rightarrow D(L_2)_p$ such that $f_1 \oplus f_2$ extends to the overlattice $L_1 \oplus_{\phi_p} L_2$.

Note that we can piece together gluing maps ϕ_p for different primes p to get a simultaneous glue map $\phi = \oplus_p \phi_p$.

The following theorem is striking in its simplicity and its consequences. It is probably known to the experts, though the author does not know a reference.

Theorem 3.8. *Let $M \oplus N \hookrightarrow L$ be a primitive extension and f_M, f_N be isometries of M and N with minimal polynomials $m(x)$ and $n(x)$. Suppose that $f = f_M \oplus f_N$ extends to L . Then*

$$dL \subseteq M \oplus N$$

where $d\mathbb{Z} = (m(x)\mathbb{Z}[x] + n(x)\mathbb{Z}[x]) \cap \mathbb{Z}$.

Proof. By definition of d we can find $u, v \in \mathbb{Z}[x]$ such that

$$d = u(x)n(x) + v(x)m(x).$$

Then $d \cdot \text{id} = u(f)n(f) + v(f)m(f)$ and further

$$\begin{aligned} dL &= (u(f)n(f) + v(f)m(f))L \\ &\subseteq u(f)n(f)L + v(f)m(f)L \\ &\subseteq \ker m(f) \oplus \ker n(f) \\ &= M \oplus N. \end{aligned}$$

In the last step we used the primitivity of $M \oplus N \hookrightarrow L$. □

Note that d divides the resultant $\text{res}(m(x), n(x))$ and both have the same prime factors. For a case where $d < \text{res}(m(x), n(x))$ consider $x^2 + 1, x^2 - 4$. We deduce the following corollary. It was originally stated in [30, Theorem 4.3] for *unimodular* primitive extensions.

Corollary 3.9. *Let M, N, L be lattices and*

$$M \oplus N \hookrightarrow L$$

a primitive extension with glue $G_M \cong G \cong G_N$. Let f_M, f_N be isometries of M and N with characteristic polynomials χ_M and χ_N . If $f_M \oplus f_N$ extends to L , then any prime dividing $|G|$ also divides the resultant $\text{res}(\chi_M, \chi_N)$.

3.2. Real orthogonal transformations and the sign invariant. In this section we review the sign invariant of a real orthogonal transformation. Proofs and details can be found in [20].

We denote by $\mathbb{R}^{p,q}$ the vector space \mathbb{R}^{p+q} equipped with the quadratic form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

Let $SO_{p,q}(\mathbb{R}) = SO(\mathbb{R}^{p,q})$ be the Lie group of real orthogonal transformations of determinant one, preserving the quadratic form. If the characteristic polynomial

$s(x)$ of $F \in SO_{p,q}(\mathbb{R})$ is of even degree $2n = p+q$ and separable, then it is reciprocal, i.e., $x^{2n}s(x) = s(x^{-1})$. It has a trace polynomial $r(x)$ defined by

$$s(x) = x^n r(x + x^{-1}).$$

Its roots are real of the form $\lambda + \lambda^{-1}$ where λ is a root of $s(x)$. Call \mathcal{T} the set of roots of $r(x)$ in the interval $(-2, 2)$. They correspond to conjugate pairs of roots $\lambda + \bar{\lambda}$ of $s(x)$ on the unit circle. We have an orthogonal direct sum decomposition

$$\mathbb{R}^{p,q} = \bigoplus_{\tau \in \mathcal{T}} E_\tau, \quad E_\tau := \ker(F + F^{-1} - \tau I).$$

On E_τ , $\tau \in \mathcal{T}$, F acts by rotation by angle $\theta = \arccos(\tau/2)$. Hence E_τ is either positive or negative definite. For $\tau \in \mathcal{T}$ this is encoded in the sign invariant.

$$\epsilon_F(\tau) = \begin{cases} +1 & \text{if } E_\tau \text{ has signature } (2, 0), \\ -1 & \text{if } E_\tau \text{ has signature } (0, 2). \end{cases}$$

Denote by $2t$ the number of roots of $s(x)$ outside the unit circle. We can recover the signature via

$$(p, q) = (t, t) + \sum_{\tau \in \mathcal{T}} \begin{cases} (2, 0) & \text{if } \epsilon_F(\tau) = +1 \\ (0, 2) & \text{if } \epsilon_F(\tau) = -1 \end{cases}$$

Two isometries $F, G \in SO_{p,q}(\mathbb{R})$ with characteristic polynomial $s(x)$ are conjugate in $O_{p,q}(\mathbb{R})$ iff $\epsilon_F = \epsilon_G$.

3.3. Lattices in number fields. In this section we review the theory of lattice isometries associated to certain reciprocal polynomials as exploited in [30]. For further reading consider [8–10].

A pair (L, f) where L is a lattice and $f \in O(L)$ an isometry with minimal polynomial $p(x)$, is called a $p(x)$ -lattice. We call two $p(x)$ -lattices (L, f) and (N, g) isomorphic if there is an isometry $\alpha : L \rightarrow N$ with $\alpha \circ f = g \circ \alpha$. Notice that this definition differs from that of McMullen in [30] where $p(x)$ is the characteristic polynomial instead.

Example 3.10. If X is a complex K3 surface and f an automorphism of X acting by multiplication with an n -th root of unity on $H^0(X, \Omega_X^2)$, then $(T(X), f)$ is a $c_n(x)$ -lattice, where $c_n(x)$ denotes the n -th cyclotomic polynomial. Note that

$$H^{2,0} \subseteq (\ker c_n(f^*|T)) \otimes_{\mathbb{Z}} \mathbb{C} \subseteq T \otimes \mathbb{C}$$

Since the kernel is defined over \mathbb{Z} , the equality $T = \ker c_n(f|T)$ follows from the minimality of T .

Given an element $a \in \mathbb{Z}[f + f^{-1}] \subseteq \text{End}(L)$ one can define a new inner product

$$\langle g_1, g_2 \rangle_a := \langle ag_1, g_2 \rangle$$

on L . We denote the resulting lattice by $L(a)$, and call this operation a twist. The pair $(L(a), f)$ is called a twisted $p(x)$ -lattice. If L is even, then so is $L(a)$.

Conversely, if we start with an irreducible, reciprocal polynomial $p(x) \in \mathbb{Z}[x]$ of degree $d = 2e$, we can associate a $p(x)$ -lattice to it as follows. Recall that $r(y)$ denotes the associated trace polynomial defined by $p(x) = x^e r(x + x^{-1})$.

Then

$$K := \mathbb{Q}[f] \cong \mathbb{Q}[x]/p(x)$$

is an extension of degree 2 of

$$k := \mathbb{Q}[f + f^{-1}] \cong \mathbb{Q}[y]/r(y)$$

with Galois involution σ defined by $f^\sigma = f^{-1}$.

Now we can define the principal $p(x)$ -lattice (L_0, f_0) by

$$L_0 := \mathbb{Z}[x]/p(x)$$

with isometry f_0 given by multiplication with x . As inner product we take

$$\langle g_1, g_2 \rangle_0 := \sum_i \frac{g_1(x_i)g_2(x_i^{-1})}{r'(x + x^{-1})} = \text{Tr}_{\mathbb{Q}}^K \left(\frac{g_1 g_2^\sigma}{r'(x + x^{-1})} \right)$$

where the sum is taken over the roots x_i of $p(x)$ and $r'(y)$ is the formal derivative of $r(y)$. This form is even with $|\det L_0| = |p(1)p(-1)|$.

The situation is particularly nice if K has class number one, $\mathbb{Z}[x]/p(x)$ is the full ring of integers \mathcal{O}_K of K and $|p(1)p(-1)|$ is square-free. In this case $p(x)$ is called a *simple reciprocal polynomial* and we get the following theorem.

Theorem 3.11. [30, 5.2] *Let $p(x)$ be a simple reciprocal polynomial, then every $p(x)$ -lattice of rank $\deg p(x)$ is isomorphic to a twist $L_0(a)$ of the principal $p(x)$ -lattice.*

Remark 3.12. If we drop the condition that $|p(1)p(-1)|$ is square-free, we have to allow twists in $r'(x+x^{-1})\mathcal{D}_K^{-1} \cap \mathcal{O}_K = 1/(x-x^{-1})\mathcal{O}_K \cap \mathcal{O}_K$, where $\mathcal{D}_K = (p'(x))\mathcal{O}_K$ is the different of K . If K/k ramifies over 2, these need not be even in general [8, §2.6]. Dropping the condition on the class number leads to so called ideal lattices surveyed in [10].

If $\mathbb{Z}[f] \cong \mathbb{Z}[x]/p(x)$ is the full ring of integers \mathcal{O}_K , then all the usual objects such as discriminant group, glue, (dual) lattice, etc. will be \mathcal{O}_K -modules.

Lemma 3.13. *Let $p(x)$ be a simple reciprocal polynomial. Then there is an element $b \in \mathcal{O}_K$ of absolute norm $|p(1)p(-1)|$ such that $L_0^\vee = \frac{1}{b}\mathcal{O}_K$. If $a \in \mathcal{O}_k$ is a twist, then*

$$L_0(a)^\vee / L_0(a) \cong \mathcal{O}_K / ab\mathcal{O}_K$$

as \mathcal{O}_K -modules.

Proof. Since $L_0^\vee \subseteq K$ is a finitely generated \mathcal{O}_K -module, it is a fractional ideal. By simplicity of $p(x)$ \mathcal{O}_K is a PID and fractional ideals are of the form $\frac{1}{b}\mathcal{O}_K$, for some $b \in \mathcal{O}_K$. Then $L_0(a)^\vee = \frac{1}{a}L_0^\vee = \frac{1}{ab}L_0$ and $D(L_0(a)) \cong \mathcal{O}_K / ab\mathcal{O}_K$. \square

Given a unit $u \in \mathcal{O}_K^\times$ and $a \in \mathcal{O}_K \setminus \{0\}$ the twist $L_0(uu^\sigma a)$ is isomorphic to $L_0(a)$ via $x \mapsto ux$ as $p(x)$ -lattice. Conversely, if $v \in \mathcal{O}_k$ and $L_0(va) \cong L_0(a)$ as $p(x)$ -lattices, then, by non-degeneracy of the trace map, we can find $u \in \mathcal{O}_K$ with $v = uu^\sigma$. Since the cokernel of the norm map $N : \mathcal{O}_K^\times \rightarrow \mathcal{O}_k^\times$ is finite, the associates of $a \in \mathcal{O}_k$ give only finitely many non-isomorphic twists.

By Lemma 3.13 the prime decomposition of $a \in \mathcal{O}_k$ in \mathcal{O}_K determines the \mathcal{O}_K -module structure of the discriminant, while twisting by a unit may change the

signature and discriminant form as follows.

Let \mathcal{T} denote the set of real places of $k = \mathbb{Q}[y]/r(y)$ that become complex in K . They correspond to the real roots τ of $r(y)$ in the interval $(-2, 2)$. Each such real place is an embedding of $\nu_\tau : k \hookrightarrow \mathbb{R}$ given by $b(y) \mapsto b(\tau)$. Its sign is recorded by $\text{sign}_\tau(b(x)) = \text{sign}(\nu_\tau(b(x)))$. We call the resulting quantity the sign invariant. If $(L_0(b), f_0)$ is a twist of the principal lattice, then (cf. [20, 4.2])

$$\epsilon_{f_0}(\tau) = \text{sign}_\tau(b/r'(y)).$$

For $n \in \mathbb{N}$ a lattice is called n -elementary if $nD_L = 0$. Let $\mathfrak{J} \subseteq \mathcal{O}_K$ be an ideal. We call a $p(x)$ -lattice \mathfrak{J} -elementary if $\mathfrak{J}D_L = 0$.

Lemma 3.14. *Let $N \hookrightarrow L$ be a primitive embedding, and consider the natural inclusion $L \hookrightarrow L^\vee$ and the projection*

$$p_N : L^\vee \rightarrow N^\vee.$$

Then there is a surjection $L^\vee/L \twoheadrightarrow N^\vee/p_N(L)$.

Proof. We have the following induced diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L^\vee & \longrightarrow & L^\vee/L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_N(L) & \longrightarrow & N^\vee & \longrightarrow & N^\vee/p_N(L) & \longrightarrow & 0 \end{array}$$

where the primitivity of $N \hookrightarrow L$ gives the surjectivity of the central vertical arrow. To see this, either use the Ext functor, or notice that a \mathbb{Z} basis of N can be completed to a basis of L . Then take the dual basis. Since the diagram is commutative, surjectivity follows. \square

We remark that if $(N, f_N) \hookrightarrow (L, f_L)$ is a primitive embedding of $p(x)$ -lattices, then the maps involved are $\mathbb{Z}[x]/p(x)$ -module homomorphisms. If L is \mathfrak{p} -elementary, then $N^\vee/G_N = p_N(L^\vee/L)$ is annihilated by \mathfrak{p} as well, or equivalently it is an $\mathcal{O}_K/\mathfrak{p}$ -vector space.

4. SMALL CYCLOTOMIC FIELDS

Motivated by the action of a non-symplectic automorphism on the transcendental lattice of a K3 surface, we study $c_n(x)$ -lattices more closely. In order to do this we review some of the general theory on cyclotomic fields. Our main reference is [53].

For $n \in \mathbb{N}$, we denote by $K = \mathbb{Q}(\zeta_n)$ the n -th cyclotomic field and by $c_n(x)$ the n -th cyclotomic polynomial. The Euler totient function $\varphi(n)$ records the degree of $c_n(x)$. The maximal real subfield of K is $k = \mathbb{Q}[\zeta_n + \bar{\zeta}_n]$. The rings of integers of these two fields are

$$\mathcal{O}_K = \mathbb{Z}[\zeta_n] \quad \text{and} \quad \mathcal{O}_k = \mathbb{Z}[\zeta_n + \bar{\zeta}_n].$$

Lemma 4.1. *The cyclotomic polynomials $c_n(x)$ are simple reciprocal polynomials for $2 \leq \varphi(n) \leq 21$, $n \neq 2^d$.*

Proof. The only non-trivial part is that the class numbers are one. This is stated in [28]. \square

Note that even though $|c_{2d}(1)c_{2d}(-1)| = 4$ is not square-free, every *even* c_{2d} -lattice ($2 \leq d \leq 5$) is a twist of the principal c_{2d} -lattice (cf. Remark 3.12).

Lemma 4.2. [53, Prop. 2.8] *If $n \in \mathbb{N}$, has two distinct prime factors, then $(1 - \zeta_n)$ is a unit in \mathcal{O}_K .*

The kernel $\mathcal{O}_k^{\times+}$ of the map

$$\text{sign} : \mathcal{O}_k^{\times} \rightarrow \{\pm 1\}^{\frac{\varphi(n)}{2}}$$

is the set of *totally positive units* of \mathcal{O}_k .

Proposition 4.3. [45, A.2] *If the relative class number $h^-(K) = h(K)/h(k)$ is odd, then $\mathcal{O}_k^{\times+} = N_k^K(\mathcal{O}_K^{\times})$.*

Corollary 4.4. *Let $n \in \mathbb{N}$ with $\varphi(n) \leq 20$. Set $K := \mathbb{Q}(\zeta_n)$ the n -th cyclotomic field. Then the group homomorphism*

$$\text{sign} : \mathcal{O}_k^{\times} / N(\mathcal{O}_K) \rightarrow \{\pm 1\}^{\frac{\varphi(n)}{2}}$$

is injective.

Proof. As $\mathbb{Q}[\zeta_n]$ is a PID for $\phi(n) \leq 20$, the relative class number is one and we may apply Proposition 4.3. \square

The first cases where the relative class number is even is for $n = 39, 56, 29$. There $h^-(\mathbb{Q}[\zeta_n]) = 2, 2, 2^3$ (cf. [53, §3]), and the sign map has a kernel of order $2, 2, 2^3$ as well. We refer the interested reader to [23, 45] for more on the relation of class numbers and totally positive units.

Proposition 4.5. *The isomorphism class of a $c_n(x)$ -lattice (T, f) , with $2 \leq \text{rk } T = \varphi(n) \leq 20$ is given by the kernel of*

$$\mathbb{Z}[x]/c_n(x) \rightarrow O(T^{\vee}/T), \quad x \mapsto \overline{f}$$

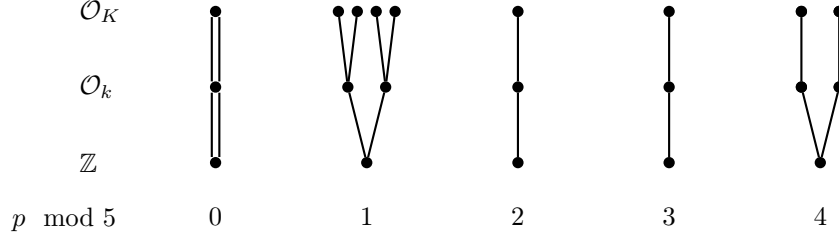
and the sign invariant of f .

Proof. By Theorem 3.11, $(T, f) \cong (L_0(ub), f_0)$. Lemma 3.13 shows that the \mathcal{O}_K -module structure of the discriminant determines the prime decomposition of $b \in \mathcal{O}_k$. By Lemma 4.4 the different isomorphism classes of $(L_0(ub), f_0)$ for fixed b and some $u \in \mathcal{O}_k^{\times}$ are determined by their sign invariant. \square

Remark 4.6. Note that we did not exclude 2-powers from this proposition.

Example 4.7. Recall that we denote by $\zeta_5 = \exp(\frac{2\pi i}{5})$ a fifth root of unity and by $c_5(x) = x^4 + x^3 + x^2 + x + 1$ its minimal polynomial. Set $K = \mathbb{Q}[\zeta_5]$ and $k = \mathbb{Q}[\zeta_5 + \zeta_5^{-1}] = \mathbb{Q}[\sqrt{5}]$. Denote their rings of integers by $\mathcal{O}_K = \mathbb{Z}[\zeta_5] = \mathbb{Z}[x]/c_5(x)$ and $\mathcal{O}_k = \mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$. Note that both are principal ideal domains. So $c_5(x)$ is simple reciprocal and the previous theorems apply. The primes factor as given in Figure 1.

Since $c_5(x)$ is a simple reciprocal polynomial, every rank 4 $c_5(x)$ -lattice is isomorphic to a twist $L_0(t), t \in \mathcal{O}_k$ of the principal $c_5(x)$ -lattice by Theorem 3.11.

FIGURE 1. Prime decompositions in $\mathbb{Q}[\zeta_5]$

The principal $c_5(x)$ -lattice L_0 has determinant $5 = |c_5(1)c_5(-1)|$. The only prime above 5 is $(x-1)$. Hence after a twist $t \in \mathcal{O}_k$

$$D(L_0(t)) = \frac{1}{(x-1)t} \mathcal{O}_K / \mathcal{O}_K \cong \mathcal{O}_K / (x-1)t$$

as \mathcal{O}_K -modules by Lemma 3.13. The \mathcal{O}_K -module structure of the discriminant group determines the prime decomposition of t up to units. The cokernel of the norm map $\mathcal{O}_K^\times \rightarrow \mathcal{O}_k^\times$ consists of 4 elements. Thus we get 4 inequivalent twists $\pm t, \pm(\zeta_5 + \zeta_5^{-1})t$ two of which have signature $(2, 2)$ but different sign invariants and the other ones signatures $(4, 0)$ and $(0, 4)$.

For later use, we note two theorems on cyclotomic polynomials.

Theorem 4.8. [1, 17] *The resultant of two cyclotomic polynomials c_m, c_n of degrees $0 < m < n$ is given by*

$$\text{res}(c_n, c_m) = \begin{cases} p^{\varphi(m)} & \text{if } n/m = p^e \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 4.9. [17, 18]

$$(\mathbb{Z}[x]c_n + \mathbb{Z}[x]c_m) \cap \mathbb{Z} = \begin{cases} p & \text{if } n/m = p^e \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$$

5. UNIQUENESS THEOREM

Proposition 5.1. *Let X be a complex K3 surface and $f \in \text{Aut}(X)$ an automorphism of finite order with $f^*(\omega) = \zeta_n \omega$ on $0 \neq \omega \in H^0(X, \Omega_X^2)$. Suppose that $\text{rk } T_X = \varphi(n)$. Then there is a unique primitive embedding*

$$T_X \hookrightarrow L_{K3}.$$

Proof. If $\varphi(n) \leq 10$, then $\text{rk } T_X + l(D_{T(X)}) \leq 2 \text{rk } T_X = 2\varphi(n) = 20$ and Theorem 3.2 provides uniqueness of the embedding. If $\varphi(n) > 10$, then $\varphi(n) \geq 12$ and ζ_n is not an eigenvalue of $f|_{\text{NS}}$. By Corollary 6.3 there are only finitely many possibilities of T_X up to isometry. Uniqueness of the embedding is checked individually in Lemma 6.4. □

Proposition 5.2. *Let X/\mathbb{C} be a K3 surface and $f \in \text{Aut}(X)$ an automorphism with $f^*(\omega) = \zeta_n^k \omega$ on $0 \neq \omega \in H^0(X, \Omega_X^2)$ where $\zeta_n = e^{2\pi i/n}$, $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. Suppose that*

- (1) $\text{rk } T_X = \varphi(n)$,
- (2) $T_X \hookrightarrow L_{K3}$ uniquely.

Then the isomorphism class of X is determined by (I, ζ_n^k) where I is the kernel of

$$\mathbb{Z}[x]/c_n(x) \rightarrow O(T^\vee/T), \quad x \mapsto \overline{f^*}.$$

Conversely, if $\zeta_X = \zeta_Y$, but $I_X \neq I_Y$, then $X \not\cong Y$.

Proof. Let X, Y be K3 surfaces and f_X, f_Y be automorphisms as in the theorem. Set $\tau = \zeta_n^k + \zeta_n^{-k}$ and $E_\tau = \ker(f|_T + f|_T^{-1} - \tau \text{id}_T)$. Looking at $\omega, \bar{\omega} \in E_\tau \otimes \mathbb{C}$ with $\omega \cdot \bar{\omega} > 0$, we see that E_τ has signature $(2, 0)$. Since the signature of T is $(2, \varphi(n) - 2)$, this determines the sign invariants of (T_X, f_X) and (T_Y, f_Y) . This is recorded by the complex n -th root of unity ζ_n^k . By assumption their discriminants have the same \mathcal{O}_K -module structure and Proposition 4.5 implies that $(T_X, f_X) \cong (T_Y, f_Y)$ as $c_n(x)$ -lattices.

Hence, we can find an isometry $\psi_T : T_X \rightarrow T_Y$ such that $f_Y \circ \psi_T = \psi_T \circ f_X$. The latter condition assures that ψ_T is compatible with the eigenspaces of f_X, f_Y . Since $\text{rk } T_X = \varphi(n)$, the eigenspaces for ζ_n^k are $H^{2,0}(X)$ and $H^{2,0}(Y)$. In particular,

$$\psi_T(H_X^{2,0}) = H_Y^{2,0}.$$

Now, choose markings ϕ_X and ϕ_Y on X and Y . They provide us with two embeddings ϕ_X and $\phi_Y \circ \psi$ of T_X into L_{K3} . By assumption (2) any two embeddings are isomorphic. That is, we can find $\psi \in O(L_{K3})$ such that the following diagram commutes.

$$\begin{array}{ccccc} T_X & \xrightarrow{f_X} & T_X \subseteq H^2(X, \mathbb{Z}) & \xrightarrow{\phi_X} & L_{K3} \\ \downarrow \psi_T & & \downarrow \psi_T & & \downarrow \exists \psi \\ T_Y & \xrightarrow{f_Y} & T_Y \subseteq H^2(Y, \mathbb{Z}) & \xrightarrow{\phi_Y} & L_{K3} \end{array}$$

By construction $\phi_Y^{-1} \circ \psi \circ \phi_X$ is a Hodge isometry. By the weak Torelli Theorem X and Y are isomorphic. Conversely, let $f_1, f_2 \in \text{Aut}(X)$ with $f_1^* \omega = f_2^* \omega = \zeta_n \omega$. Since the image of $\text{Aut}(X) \rightarrow O(T)$ is a cyclic group, $f_1|_T = f_2|_T$. In particular $I_1 = I_2$. \square

Remark 5.3. Replacing f by a power f^k with k coprime to n , we can fix the action on the 2-forms. This corresponds to the Galois action $\zeta_n \mapsto \zeta_n^k$ on $\mathbb{Q}(\zeta_n)$. In case the embedding is not unique, one can fix the isometry class of NS. Then isomorphism classes of primitive embeddings with $T^\perp = \text{NS}$ are given by glue maps $\phi : T^\vee/T \rightarrow \text{NS}^\vee/\text{NS}$ with $-q_T = q_{\text{NS}} \circ \phi$ modulo the action of $O(\text{NS})$ on the right. We can also allow for an action of the centralizer of $f|_T$ in $O(T)$ on the left. It should be noted that the proposition can be applied to automorphisms of infinite order too.

Theorem 5.4. *Let $X_i, i = 1, 2$ be complex K3 surfaces and $f_i \in \text{Aut}(X_i)$ automorphisms of finite order with $f_i^*(\omega_i) = \zeta_n \omega_i$ on $0 \neq \omega_i \in H^0(X_i, \Omega_{X_i}^2)$ such that $\text{rk } T(X_i) = \varphi(n)$. Then $X_1 \cong X_2$ if and only if $I_1 = I_2$ where I_i is the kernel of*

$$\mathbb{Z}[x]/c_n(x) \rightarrow O(\text{NS}(X_i)^\vee/\text{NS}(X_i)), \quad x \mapsto \overline{f_i^*}.$$

Theorem 5.5. *Let $X_i/k, i = 1, 2$ be K3 surfaces over an algebraically closed field of characteristic not 2 or 3. Let f_i be tame automorphisms with $f_i^* \omega_i = \zeta_n \omega_i \in H^0(X_i, \Omega_{X_i}^2)$, and $\varphi(n) = 22 - \text{rk NS}(X_i)$. If $I_1 \cong I_2$, then $X_1 \cong X_2$ where I_i is the kernel of*

$$\mathbb{Z}[x]/c_n(x) \rightarrow O(\text{NS}(X_i)^\vee / \text{NS}(X_i)), \quad x \mapsto \overline{f_i^*}.$$

Proof. In the characteristic 0 case we can work over \mathbb{C} . In the tame case we can lift (X, NS, f) [22, Thm. 3.2]. This preserves I , and since f is tame, there is a unique lift of ζ_n to an n -th root as well. We can apply the previous Theorem 5.4. \square

6. VORONTSOV'S THEOREM

In this section we give a generalization as well as a (new) uniform proof of Vorontsov's Theorem and related results using the uniqueness theorem of this paper. On the way we can correct results in [21, 48]. Originally this section was only concerned with a new proof of Vorontsov's Theorem. Then I heard of the work of S. Taki. Among others he started the classification of purely non symplectic automorphisms of order n on complex K3 surfaces such that $\varphi(n) \geq 12$. It was then that I realized that the language developed here is suitable for a generalization of Vorontsov's theorem.

Let X be a complex K3 surface and

$$\rho : \text{Aut}(X) \rightarrow O(\text{NS}(X))$$

be the representation of the automorphism group of X on the Néron-Severi group. Set $H(X) := \ker \rho$ and $h(X) := |H(X)|$ its order. Nikulin [36] showed that $H(X)$ is a finite cyclic group and $\varphi(h(X)) \mid \text{rk } T(X)$. Vorontsov's Theorem looks at the extremal case $\varphi(h(X)) = \text{rk } T(X)$. That is, X has an automorphism of (maximal) order $h(X)$ acting trivially on $\text{NS}(X)$.

Theorem 6.1. [25, 27, 41, 52] *Set $\Sigma := \{66, 44, 42, 36, 28, 12\}$ and $\Omega := \{3^k \ (1 \leq k \leq 3), 5^l \ (l = 1, 2), 7, 11, 13, 17, 19\}$.*

- (1) *Let X/\mathbb{C} be a K3 surface with $\varphi(h(X)) = \text{rk } T(X)$. Then $h(X) \in \Sigma \cup \Omega$. The transcendental lattice $T(X)$ is unimodular iff $h(x) \in \Sigma$.*
- (2) *Conversely, for each $N \in \Sigma \cup \Omega$, there exists, modulo isomorphisms, a unique K3 surface X/\mathbb{C} such that $h(X) = N$ and $\varphi(h(X)) = \text{rk } T(X)$. Moreover, $T(X)$ is unimodular iff $N \in \Sigma$.*

For our alternative proof, we first show that h and the conditions given determine the transcendental lattice (T, f) as a $c_h(x)$ -lattice (up to powers of f) and that T embeds uniquely into the K3-lattice. Then Theorem 5.4 provides the uniqueness. Next we show that for each $h \in \Sigma \cup \Omega$ we can find (T, f) such that f is of order h and acts trivially on T^\vee/T . This f can be glued to the identity, which trivially preserves an ample cone. Then the strong Torelli theorem provides the existence. Alternatively, the equations of X and f are well known and found in Tables 2, 3.

Proposition 6.2. *Let f be a non-symplectic automorphism acting with order n on the global 2-forms of a complex K3 surface X with $\varphi(n) = \text{rk } T(X)$. Then*

$$\det T(X) \mid \text{res}(c_n, \mu)$$

where $\mu(x)$ is the minimal polynomial of $f|_{\text{NS}(X)}$. If $\text{res}(c_n, \mu) \neq 0$ and f is purely non-symplectic, then T is n -elementary, i.e. $nD_T = 0$.

Proof. If c_n and μ have a common factor, then $\text{res}(c_n, \mu) = 0$ and the statement is certainly true. We may assume that $\gcd(c_n, \mu) = 1$. Then we know that

$$T = T(X) = \ker c_n(f|_{H^2(X, \mathbb{Z})})$$

and we can view (T, f) as a $c_n(x)$ -lattice. Then $D_T \cong \mathcal{O}_K/I$, $K = \mathbb{Q}(\zeta_n)$, for some ideal $I < \mathcal{O}_K$. The isomorphisms $D_{\text{NS}} \cong D_T \cong \mathcal{O}_K/I$ are compatible with f . In particular $\mu(f|_{\text{NS}}) = 0$ implies that $\mu(\bar{f}|_{D_T}) = 0$, i.e., $\mu(\zeta_n) \in I$. By definition of norm and resultant

$$|\det T(X)| = |\mathcal{O}_K/I| = N(I) \mid N(\mu(\zeta_n)) = \text{res}(c_n, \mu).$$

It remains to prove that $I \mid n$. Since for $\varphi(n) \leq 20$ there is only a single prime ideal above $p \mid n$ in $\mathbb{Q}[\zeta_n + \zeta_n^{-1}]$, it suffices to check that

$$\text{res}(c_n, \mu) \mid \text{res}\left(c_n, \prod_{k < n} c_k\right) \mid N(n) = n^{\varphi(n)}.$$

This is easily seen with Theorem 4.8. \square

Corollary 6.3. *Suppose that $\text{rk } T(X) = \varphi(n)$ and f is purely non-symplectic with $\gcd(\mu_{f|_{\text{NS}}, c_n}) = 1$. Then we have the following restrictions on $T(X)$*

- (0) T has signature $(2, \varphi(n) - 2)$;
- (1) $2 \leq \varphi(n) \leq 21$;
- (2)

$$(x - 1) \mid \mu \quad \text{and} \quad \mu \mid \prod_{k < n} c_k$$

where $\deg \mu \leq 22 - \varphi(n)$ and $\det T \mid \text{res}(c_n, \mu)$;

- (3) $\exists b \in \mathcal{O}_k$ such that $T \cong L_0(b)$ is a twist of the principal $c_n(x)$ -lattice.

The resulting determinants are listed in Table 1.

Proof. (0) and (1) are clear. (2) Since f is of finite order, $\mu = \mu(f|_{\text{NS}})$ is separable and we apply Proposition 6.2. (3) By assumption T is a $c_n(x)$ -lattice of rank $\deg c_n$. For $2 \leq \varphi(n) \leq 21$ all cyclotomic polynomials $c_n(x)$ are simple and Theorem 3.11 provides the claim.

It remains to compute the values of Table 1. We shall do the computation for $n = 28$. The other cases are similar. By Theorem 4.8 a factor $c_k(x)$ of $\mu(x)$ will contribute to the resultant if and only if n/k is a prime power. Hence, the only possibilities are $c_4, c_7, c_{14}, c_4c_7, c_4c_{14}$ which are of degree 2, 6, 6, 8, 8 and give resultants $7^2, 2^6, 2^6, 2^67^2, 2^67^2$. The principal $c_{28}(x)$ -lattice is unimodular and up to units there is only a single twist above 2 of norm 2^6 and a single twist above 7 of norm 7^2 . This results in the 4 possible determinants $1, 2^6, 7^2, 2^67^2$. We can exclude 7^2 and 2^67^2 since there is no twist of the right signature $(2, 10)$. This leaves us with determinants 1 and 2^6 . \square

TABLE 1. Possible determinants of the transcendental lattice

n	$\varphi(n)$	$\det T$	n	$\varphi(n)$	$\det T$
3, 6	2	3	20	8	$2^4, 2^4 5^2$
4	2	2^2	21, 42	12	$1, 7^2$
5, 10	4	5	24	8	$2^2, 2^6, 2^2 3^4, 2^6 3^4$
7, 14	6	7	25, 50	20	5
8	4	$2^2, 2^4$	27, 54	18	$3, 3^3$
9, 18	6	$3, 3^3$	28	12	$1, 2^6$
11, 22	10	11	32	16	$2^2, 2^4, 2^6$
12	4	$1, 2^2 3^4, 2^4$	33, 66	20	1
13, 26	12	13	36	12	$1, 3^4, 2^6 3^2$
15, 30	8	$5^2, 3^4$	40	16	2^4
16	8	$2^2, 2^4, 2^6, 2^8$	44	20	1
17, 34	16	17	48	16	2^2
19, 38	18	19	60	16	—

Lemma 6.4. *Let $2 \leq \varphi(n) \leq 20$. For each $p \mid n$, there is a unique prime ideal \mathcal{O}_k above p . In particular the $c_n(x)$ -lattices T of Table 1 are determined up to isomorphism by their determinants. They admit a unique primitive embedding into L_{k3} , except $(n, \det T) = (32, 2^6)$ which does not embed in L_{k3} .*

Proof. The c_n -lattices are twists of the principal c_n -lattice. Twists correspond to ideals in \mathcal{O}_k which are well known from the theory of cyclotomic fields.

Since $\varphi(n) + l(T^\vee/T) \leq 20$ for all pairs (n, d) except $(25, 5)$, $(27, 3^3)$ and $(32, 2^6)$ Theorem 5.4 provides uniqueness (and existence) of a primitive embedding outside those cases.

We have to check in case $n = 25$ that T embeds uniquely into the K3-lattice. It has rank 20 and determinant 5. Its orthogonal complement NS is an indefinite lattice of determinant 5. It is unique in its genus and the canonical map $O(\text{NS}) \rightarrow O(\text{NS}^\vee/\text{NS})$ is surjective since both groups are generated by $-id$. By [37, 1.14.1] the embedding of T into L_{k3} is unique.

For the case $(27, 3^3)$, we need more theory not explained here, see e.g. [33]. By [33, VIII 7.6] NS is 3-semiregular and p -regular for $p \neq 3$. Now [33, VIII 7.5] provides surjectivity of $O(\text{NS}) \rightarrow O(\text{NS}^\vee/\text{NS})$ and uniqueness in its genus (alternatively cf. [31, 32]). Uniqueness of the embedding follows again with [37, 1.14.1].

It remains to check that $(32, 2^6)$ does not embed into the K3-lattice. Suppose that it does. Then its orthogonal complement is isomorphic to $A_1(-1) \oplus 5A_1$ which is the only lattice of signature $(1, 5)$ and discriminant group \mathbb{F}_2^6 (cf. [14, Tbl. 15.5]). Its discriminant form takes half integral values. Up to sign it is isomorphic to the discriminant form of its orthogonal complement $T \cong U(2) \oplus U(2) \oplus D_4 \oplus E_8$ which takes integral values, contradicting the existence of a primitive embedding. \square

Proof of Theorem 6.1. Let X be as in the theorem and f be a generator of $H(X)$, that is, $f|NS = id$ and (T, f) is a simple $c_n(x)$ -lattice. As usual the discriminant group is a finite \mathcal{O}_K -module, and we can find an ideal $I < \mathcal{O}_K$ such that

$$NS^\vee/NS \cong T^\vee/T \cong \mathcal{O}_K/I.$$

The isometry f acts via multiplication by x on the right hand side. The condition that it acts trivially on NS translates to

$$(x - 1) \in I.$$

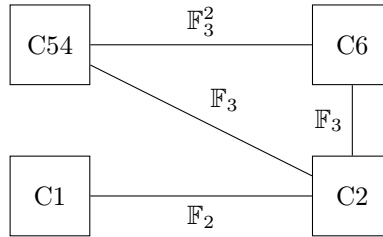
- (1) If n has distinct prime factors, then, by Lemma 4.2, $(x - 1)$ is a unit. Hence, $I = \mathcal{O}_K$ and T is unimodular.
- (2) If $n = p^k$, then \mathcal{O}_K is totally ramified over p and $(x - 1)$ is prime of norm p . In particular, either $I = (x - 1)\mathcal{O}_K$, or T is unimodular.

Collecting the entries (n, d) with $d = 1$ and n even from Table 1 leads to Σ , while (2) leads to (p^k, p) , i.e., Ω . Now Lemma 6.4 and Theorem 5.4 provide uniqueness of the K3 surface up to isomorphism. Note that the 2-power entries in Table 1 do not satisfy (2). Instead of isolated examples there are only families with trivial action (see [44]).

For the existence part, note that for each $n \in \Sigma \cup \Omega$ there is a $c_n(x)$ -lattice (T, f) of signature $(2, \varphi(n) - 2)$ with trivial action on the discriminant group. It embeds primitively into the K3-lattice and we can glue f to the identity on the orthogonal complement. Then the strong Torelli theorem and the surjectivity of the period map provide the existence of the desired K3 surface and its automorphism. \square

Lemma 6.5. *The pair $(54, 3^3)$ is not realized by a K3 surface.*

Proof. Suppose that (X, f) realizes $(54, 3^3)$. Then by Corollary 6.3 the characteristic polynomial of $f|H^2(X, \mathbb{Z})$ is $c_{54}c_6c_2c_1$. The resultants look as follows: $\text{res}(c_{54}, c_6) = 3^2$, $\text{res}(c_6, c_2) = 3$, $\text{res}(c_2, c_1) = 2$. A similar reasoning as in Proposition 6.2 for each factor c_i yields a unique gluing diagram for $C_i = \ker c_i(f|H^2(X, \mathbb{Z}))$. Here an edge symbolizes a glue map. Associated to an edge is the glue G .



In particular this determines the lattices $C2$ and $C1$ with Gram matrices (-18) and (2) . There is a unique gluing of these lattices computed in Example 3.5. It results in the lattice

$$C1C2 = \ker c_1c_2(f|H^2(X, \mathbb{Z})) \cong \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}.$$

We know the determinant of $C6$ is 3^3 . Twisting the principal c_6 -lattice results in $C6 = A_2(3)$. The gluing of $C1C2$ and $C6$ equals $NS(X)$. It has discriminant group \mathbb{F}_3^3 . The lattices $C1C2$ and $C6$ are glued over \mathbb{F}_3 . Their discriminant groups are

$D_1 = \mathbb{Z}/9\mathbb{Z}$ and $D_2 = \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{F}_3$. The gluing must result in a 3-elementary lattice. We can apply Lemma 3.14 to get that the glue $3D_1 = G_1 \cong G_2 = 3D_2$. This gluing is uniquely determined (up to ± 1) and a quick calculation shows that the resulting lattice is not 3-elementary. \square

TABLE 2. Realized determinants in ascending order of $\varphi(n) \leq 10$

n	$\det T$	X	f	
3, 6	3	$y^2 = x^3 - t^5(t-1)^5(t+1)^2$	$(\zeta_3 x, \pm y, t)$	[25]
4	2^2	$y^2 = x^3 + 3t^4x + t^5(t^2 - 1)$	$(-x, \zeta_4 y, -t)$	
5, 10	5	$y^2 = x^3 + t^3x + t^7$	$(\zeta_5^3 x, \pm \zeta_5^2 y, \zeta_5^2 t)$	[25]
8	2^2	$y^2 = x^3 + tx^2 + t^7$	$(\zeta_8^6 x, \zeta_8 y, \zeta_8^6)$	[44]
	2^4	$t^4 = (x_0^2 - x_1^2)(x_0^2 + x_1^2 + x_2^2)$	$(\zeta_8 t; x_1 : x_0 : x_2)$	
12	1	$y^2 = x^3 + t^5(t^2 - 1)$	$(-\zeta_3 x, \zeta_4 y, -t)$	[25]
	$2^2 3^2$	$y^2 = x^3 + t^5(t^2 - 1)^2$	$(-\zeta_3 x, \zeta_4 y, t)$	
	2^4	$y^2 = x^3 + t^5(t^2 - 1)^3$	$(-\zeta_3 x, \zeta_4 y, -t)$	
7, 14	7	$y^2 = x^3 + t^3x + t^8$	$(\zeta_7^3 x, \zeta_7 y, \zeta_7^2 t)$	[25]
9, 18	3	$y^2 = x^3 + t^5(t^3 - 1)$	$(\zeta_9^2 x, \pm \zeta_9^3 y, \zeta_9^3 t)$	[25]
	3^3	$y^2 = x^3 + t^5(t^3 - 1)^2$	$(\zeta_9^2 x, \pm y, \zeta_9^3 t)$	
16	2^2	$y^2 = x^3 + t^2x + t^7$	$(\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^{10} t)$	[43, 4.2]
	2^4	$y^2 = x^3 + t^3(t^4 - 1)x$	$(\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$	[15, 4.1]
	2^6	$y^2 = x^3 + x + t^8$	$(-x, iy, \zeta_{16} t)$	[43, 2.2]
20	2^4	$y^2 = x^3 + (t^5 - 1)x$	$(-x, \zeta_4 y, \zeta_5 t)$	
	$2^4 5^2$	$y^2 = x^3 + 4t^2(t^5 + 1)x$	$(-x, \zeta_4 y, \zeta_5 t)$	
24	2^2	$y^2 = x^3 + t^5(t^4 + 1)$	$(\zeta_3 \zeta_8^6 x, \zeta_8 y, \zeta_8^2 t)$	
	2^6	$y^2 = x^3 + (t^8 + 1)$	$(\zeta_3 x, y, \zeta_8 t)$	
	$2^2 3^4$	$y^2 = x^3 + t^3(t^4 + 1)^2$	$(\zeta_3 \zeta_8^6 x, \zeta_8 y, \zeta_8^6 t)$	
	$2^6 3^4$	$y^2 = x^3 + x + t^{12}$	$(-x, \zeta_{24}^6 y, \zeta_{24} t)$	
15, 30	5^2	$y^2 = x^3 + 4t^5(t^5 + 1)$	$(\zeta_3 x, \pm y, \zeta_5 t)$	
	3^4	$y^2 = x^3 + t^5x + 1$	$(\zeta_{15}^{10} x, \pm y, \zeta_{15} t)$	
11, 22	11	$y^2 = x^3 + t^5x + t^2$	$(\zeta_{11}^5 x, \pm \zeta_{11}^2 y, \zeta_{11}^2 t)$	[25]

Theorem 6.6. *Let X be a K3 surface and f a purely non-symplectic automorphism of order n such that $\text{rk } T = \varphi(n)$ and ζ_n is not an eigenvalue of $f|_{\text{NS} \otimes \mathbb{C}}$.*

Set $d = |\det \text{NS}|$, then X is determined up to isomorphism by the pair (n, d) . Conversely, all possible pairs (n, d) and equations for X and f are given in Tables 2, 3.

Proof. Comparing Tables 1 and 2, 3 we have to exclude the pairs $(16, 2^8)$ and $(54, 3^3)$. For $(16, 2^8)$, this is done in [43, 4.1]. The pair $(54, 3^3)$ is ruled out in Lemma 6.5.

TABLE 3. Purely non-symplectic automorphisms with $\varphi(n) \geq 10$

n	$\det T$	X	f	
13, 26	13	$y^2 = x^3 + t^5x + t$	$(\zeta_{13}^5x, \pm\zeta_{13}y, \zeta_{13}^2t)$	[25]
21, 42	1	$y^2 = x^3 + t^5(t^7 - 1)$	$(\zeta_{42}^2x, \zeta_{42}^3y, \zeta_{42}^{18}t)$	[25]
21, 42	7^2	$y^2 = x^3 + 4t^4(t^7 - 1)$	$(\zeta_3\zeta_7^6x, \pm\zeta_7^2y, \zeta_7t)$	
21, 42	7^2	$y^2 = x^3 + t^3(t^7 + 1)$	$(\zeta_3\zeta_7^3x, \pm\zeta_7y, \zeta_7t)$	
21	7^2	$x_0^3x_1 + x_1^3x_2 + x_0x_2^3 - x_0x_3^3$	$(\zeta_7x_0 : \zeta_7x_1 : x_2 : \zeta_3x_3)$	
28	1	$y^2 = x^3 + x + t^7$	$(-x, \zeta_4y, -\zeta_7t)$	[25]
	2^6	$y^2 = x^3 + (t^7 + 1)x$	$(-x, \zeta_4y, \zeta_7t)$	
	2^6	$y^2 = x^3 + (t^7 + 1)x$	$(x - (y/x)^2, \zeta_4(y - (y/x)^3), \zeta_7t)$	
17, 34	17	$y^2 = x^3 + t^7x + t^2$	$(\zeta_{17}^7x, \zeta_{17}y, \zeta_{17}^2t)$	[25]
34	17	$x_0x_1^5 + x_0^5x_2 + x_1^2x_2^4 = y^2$	$(-y; x_0 : \zeta_{17}x_1, \zeta_{17}^5x_2)$	
32	2^2	$y^2 = x^3 + t^2x + t^{11}$	$(\zeta_{32}^{18}x, \zeta_{32}^{11}y, \zeta_{32}^2t)$	[40]
	2^4	$y^2 = x_0(x_1^5 + x_0^4x_2 + x_1x_2^4)$	$(\zeta_{32}y; \zeta_{32}^2x_0 : x_1 : \zeta_{32}^{24}x_2)$	
36	1	$y^2 = x^3 - t^5(t^6 - 1)$	$(\zeta_{36}^2x, \zeta_{36}^3y, \zeta_{36}^{30}t)$	[25]
	3^4	$y^2 = x^3 + x + t^9$	$(-x, \zeta_4y, -\zeta_9t)$	
	2^63^2	$x_0x_3^3 + x_0^3x_1 + x_1^4 + x_2^4$	$(x_0 : \zeta_9^3x_1 : \zeta_4\zeta_9^3x_2 : \zeta_9x_3)$	
40	2^4	$z^2 = x_0(x_0^4x_2 + x_1^5 - x_2^5)$	$(x_0 : \zeta_{20}x_1 : \zeta_4x_2; \zeta_8z)$	[27]
48	2^2	$y^2 = x^3 + t(t^8 - 1)$	$(\zeta_{48}^2x, \zeta_{48}^3y, \zeta_{48}^6t)$	[27]
19, 38	19	$y^2 = x^3 + t^7x + t$	$(\zeta_{19}^7t, \zeta_{19}y, \zeta_{19}^2t)$	[25]
27, 54	3	$y^2 = x^3 + t(t^9 - 1)$	$(\zeta_{27}^2x, \zeta_{27}^3y, \zeta_{27}^6t)$	[25]
27	3^3	$x_0x_3^3 + x_0^3x_1 + x_2(x_1^3 - x_2^3)$	$(x_0 : \zeta_{27}^3x_1 : \zeta_{27}^{21}x_2 : \zeta_{27}x_3)$	
25, 50	5	$z^2 = (x_0^6 + x_0x_1^5 + x_1x_2^5)$	$(z; x_0 : \zeta_{25}^5x_1 : \zeta_{25}^4x_2)$	[25]
33, 66	1	$y^2 = x^3 + t(t^{11} - 1)$	$(\zeta_{66}^2x, \zeta_{66}^3y, \zeta_{66}^6t)$	[25]
44	1	$y^2 = x^3 + x + t^{11}$	$(-x, \zeta_4y, \zeta_{11}, t)$	[25]

By Lemma 6.4 the transcendental lattice is uniquely determined by (n, d) and embeds uniquely into L_{K3} . By Theorem 5.4 X is determined up to isomorphism by (ζ_n, I) , where I is the kernel of

$$\mathbb{Z}[x]/c_n(x) \rightarrow O(T^\vee/T), \quad x \mapsto \overline{f}$$

and $f^*\omega_X = \zeta_n\omega_X$ for $0 \neq \omega_X \in H^0(X, \Omega_X^2)$. By Lemma 6.4, I is determined uniquely by (n, d) . Replacing f with f^k , $(n, k) = 1$, does not affect (n, d) , hence I . However, in this way we can fix a primitive n -th root of unity ζ_n .

It remains to compute the Néron-Severi group of the examples in Tables 2, 3 not found in the literature. In most cases this can be done by collecting singular fibers of an elliptic fibration or determining the *fixed lattice* $S(f^k) = H^2(X, \mathbb{Z})^f$ of a suitable power of the automorphism f through its fixed points. The corresponding tables of fixed lattices are collected in [4].

(4, 2²) We see two fibers of type II^* over $t = 0, \infty$ and two fibers of type I_2 over $t = \pm 1$. Then $\text{NS} \cong U \oplus 2E_8 \oplus 2A_1$ as expected. The two form is given in local coordinates by $dx \wedge dt/2y$, and $f^*(dx \wedge dt/2y) = -dx \wedge -dt/(2y\zeta_4) = \zeta_4^3 dx \wedge dt/2y$. Hence the action is non-symplectic. The fixed lattice is $U \oplus 2E_8$ while the I_2 fibers are exchanged. Giving that $f|_{\text{NS}}$ has order two.

(8, 2⁴) The fourfold cover of \mathbb{P}^2 is a special member of a family in [3, Ex. 5.3]. It has five A_3 singularities. The fixed locus of the non-symplectic involution f^4 consists of 8 rational curves, where each A_3 configuration contains 1 fixed curve. Hence, its fixed lattice is of rank 18 and determinant -2^4 . It equals NS .

(12, 2⁴) We get fibers of type $1 \times II, 1 \times II^*, 2 \times I_0^*$ which results in the lattice $\text{NS} = U \oplus E_8 \oplus 2D_4$.

(12, 2²3²) This time zero section and fibers span the lattice $\text{NS} = U \oplus E_8 \oplus 2A_2 \oplus D_4$.

(15, 5²) This elliptic K3 surface arises as a degree 5 base change from the rational surface $Y : y^2 = x^3 + 4t(t+1)$. We see the section $(x, y) = (\zeta_3^k, 1 + 2t)$ of Y and then $(x, y) = (\zeta_3^k, 1 + 2t^5)$ generating the Mordell-Weil group of $X_{(15, 5^2)}$. Alternatively one can compute that NS is the fixed lattice of f^3 .

(15, 3⁴) The 5-th power f^5 of f is a non-symplectic automorphism of order 3 acting trivially on NS . It has 2 fixed curves of genus 0 lying in the E_7 fiber and 6 isolated fixed points over $t = 0$ and $t = \infty$. The classification of the fixed lattices of non-symplectic automorphisms of order 3 provides the fixed lattice of f^5 which equals NS . In order to get explicit generators of the Mordell-Weil group we can base change with $t \mapsto t^3$ from the rational surface $y^2 = 2x^3 + tx + t^4$ with sections $(x, y) = (t, t^2 + t), (0, t^2)$.

(34, 17) In the first case the fixed locus of g_1^{17} consists of a curve ($y = 0$) of genus 8 and a rational curve - the zero section. This leads to the fixed lattice $U \oplus 2(-2)$.

Since the fixed locus of g_2^{17} is a curve of genus 8, $S(g_1^{17}) \cong (2) \oplus 2(-2)$. Note that there is an A_4 singularity at zero. Since the fixed lattices of the two automorphisms are different, the actions are distinct as well.

(20, 2⁴) The elliptic fibration has 5 fibers of type III and a single fiber of type III^* . This results in the lattice $U \oplus E_7 \oplus 5A_1$ spanned by fiber components and the zero section. It has determinant 2^6 . Since there is also a 2-torsion section, $\det \text{NS} = 2^4$.

(20, 2⁴5²) In this case $X_{(20, 2^4 5^2)}$ has a single fiber of type I_0^* and 6 fibers of type III . This results in the lattice $U \oplus D_4 \oplus 6A_1$ of rank 12 and determinant 2^8 . Again there is 2-torsion. We reach a lattice of determinant 2^6 . We get X by a degree 5 base change from $y^2 = x^3 + 4t^2(t+1)x$ with section $(x, y) = (t^2, t^3 + 2t^2)$. We get the sections $(x, y) = (t^6, t^9 + 2t^4)$ and $(-x(t), iy(t))$ generating the Mordell-Weil lattice $2A_1^\vee(5)$ of X .

(21, 7²) We can base change this elliptic fibration from $y^2 = x^3 + 4t^4(t - 1)$ to get the sections. $(x, y) = (\zeta_3^k t^6, 2t^2 + t^9)$ generating the Mordell-Weil lattice of X . To double check note that f^3 is an order 7 non-symplectic automorphism acting trivially on NS and not fixing a curve of genus 0 point-wise. There is only a single possible fixed lattice of rank 10, namely $U(7) \oplus E_8$. For the other possible action see Lemma 7.23.

(24, 2²) The fibration has 4 type II fibers, one type I_0^* and an II^* fiber. We get $NS = U \oplus D_4 \oplus E_8$.

(24, 2⁶) In this case the fixed locus of f^{12} consists of 4 rational curves and a curve of genus 1. This leads to a fixed lattice of rank 14 and determinant 2^6 as expected.

(24, 2²3⁴) The trivial lattice is $U \oplus D_4 \oplus 4A_2$. It equals NS for absence of torsion sections.

(24, 2⁶3⁴) Since X has a purely non-symplectic automorphism of order 24, the rank of NS is either 6 or 14. $\text{Fix}(f^{12})$ consists of 2 smooth curves of genus 0. Its fixed lattice $S(f^{12})$ is 2-elementary of rank 10 and determinant -2^8 . We see that $\text{rk NS}(X) = 14$. Since the orthogonal complement of $S = S(f^{12})$ is of rank 4, the glue D_S is at most 4 dimensional subspace. Then $2^4 \leq |D_S/G_S| \leq |D_{NS}|$ by 3.4. Hence $2^4 \mid \det NS$. Note that $S(f^{12}) = \ker f^{12} - 1$ and then $\ker c_{24}c_8(f) = S(f^{12})^\perp$ is of rank 12. This shows that the characteristic polynomial of $f|NS$ is divided by c_8 but not by c_{24} . We are in the situation of the theorem. As $2^4 \mid \det NS$, it is either -2^6 or $-2^6 3^4$. We show that $3 \mid \det NS(X)$. A computation reveals that $\text{Fix}(f^8)$ consists of a smooth curve of genus 1 and 3 isolated fixed points. This leads to a fixed lattice

$$S(f^8) = \ker(c_8 c_4 c_2 c_1)(f) \cong U(3) \oplus 3A_2$$

of rank 8 and determinant -3^5 . Now we view $S(f^8)$ as a primitive extension of $\ker c_8(f) \oplus \ker c_4 c_2 c_1(f)$. The rank of both summands is 4, while the length of the discriminant group of $S(f^8)$ is 5. Then each summand must contribute to the discriminant group. We see that $3 \mid \det \ker c_8(f)$. However,

$$3 \nmid \text{res}(c_8, c_{12}c_6c_4c_3c_2c_1) = 2^4.$$

In particular the 3 part of $D(\ker c_8(f))$ cannot be glued inside NS. Then $3 \mid \det NS$.

(27, 3³) The action of f^9 has an isolated fixed point and a fixed curve of genus 3. We see that the fixed lattice of f^9 is $U(3) \oplus A_2 = NS$. It is spanned by the 4 lines at $x_3 = 0$. Note that f^3 acts trivially on NS while f does not.

(28, 2⁶) This fibration has 8 fibers of type III and a 2-torsion section. Together they generate the Néron-Severi group.

(32, 2²) The elliptic fibration has a singular fiber of type I_0^* , of type II and 16 of type I_1 . Thus $NS \cong U \oplus D_4$. Here f has 6 isolated fixed points.

(32, 2⁴) The fixed locus of $S(f^{16})$ is the strict transform of $y = 0$ which is the disjoint union of a rational curve and a curve of genus 5. Thus $\det NS = 2^4$. Note that f has 4 isolated fixed points.

(36, 3⁴) The fixed curves of f^{12} are a smooth of genus 0 over $t = 0$ and the central rational curve in the D_4 fiber. This leads to the fixed lattice $U \oplus 4A_2 = NS$.

(36, 2⁶3²) If we can show that $2 \mid \det NS$, then the only possibility is $\det NS = -2^6 3^2$. The action of f^{18} fixes a smooth curve of genus 3 and nothing else. Hence, its fixed lattice S is 2-elementary of rank 8 and determinant 2^8 . Denote by $C = S^\perp \subseteq NS$ the orthogonal complement of S inside NS . It has rank 2. Assume that $2 \nmid \det NS$. Then $D(S)_2 \cong D(C)_2$ which is impossible, since $D(S)_2$ has dimension 8, while $D(C)_2$ is generated by at most 2 elements. \square

Remark 6.7. The pair $(21, 7^2)$ contradicts the main result of J. Jang in [21, 2.1]. There it is claimed that a purely non-symplectic automorphism of order 21 acts trivially on NS . As a consequence it is claimed that there is only a single K3 surface of order 21. However there are two. The pair $(28, 2^6)$ and its uniqueness are probably known to J. Jang independently.

The pair $(32, 2^4)$ contradicts the main result of S. Taki in [49]. There the uniqueness of $(X, \langle g \rangle)$ where g is a purely non-symplectic automorphism of order 32 is claimed.

In [48, 1.8, 4.8] S. Taki classifies non symplectic automorphisms of 3-power order acting trivially on NS . The author is missing a case. It is claimed that if $NS(X) = U(3) \oplus A_2$ then there is no purely non-symplectic automorphism of order 9 acting trivially on NS . The pair $(27, 3^3)$ contradicts this result - there the automorphism acts with order 3 on NS . It is a special member of the family

$$x_0 x_3^3 + x_0^3 x_1 + x_2(x_1 - x_2)(x_1 - ax_2)(x_1 - bx_2)$$

with automorphism given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \zeta_9^3 x_1 : \zeta_9^3 x_2 : \zeta_9 x_3)$ and generically trivial action on NS as a fixed point argument shows. It was found by first computing the action of f on cohomology through gluing. Thus proving its existence and then specializing a family with automorphism of order 3 given in [2, 4.9].

7. CLASSIFICATION OF PURELY NON-SYMPLECTIC AUTOMORPHISMS OF HIGH ORDER

Let $X_i, i = 1, 2$, be K3 surfaces and $G_i \subset \text{Aut}(X_i)$ subsets of their automorphism groups. We say that the pairs $(X_1, G_1) \cong (X_2, G_2)$ are isomorphic if there is an isomorphism $\phi : X_1 \rightarrow X_2$ with $\phi \circ G_1 = G_2 \circ \phi$.

Theorem 7.1. *Let X be a K3 surface and $\mathbb{Z}/n\mathbb{Z} \cong G \subseteq \text{Aut}(X)$ a purely non-symplectic subgroup with $\varphi(n) \geq 12$. All possible pairs are found in Table 3.*

Theorem 7.2. *Let $X_{(n,d)}$ be as in Tables 2, 3.*

- (1) For $(n, d) = (66, 1), (44, 1), (50, 5), (42, 1), (28, 1), (36, 1), (32, 2^2), (32, 2^4), (40, 2^4), (54, 3), (27, 3^3), (24, 2^2), (16, 2^2)$, we have

$$\text{Aut}(X_{(n,d)}) = \langle g_{(n,d)} \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

- (2) For $(n, d) = (28, 2^6), (12, 1), (16, 2^4), (20, 2^4)$ we have

$$\text{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Remark 7.3. In [27] (1) is proven for $n = 66, 44, 50$ and 40 by a different method. By the classification in [38], all other entries of Tables 2 and 3 have infinite automorphism group.

Before proving the theorems we refine our terminology. A g -lattice is a pair (A, a) , where A is a lattice and $a \in O(A)$ an isometry. A morphism $\phi : (A, a) \rightarrow (B, a)$ of g -lattices is an isometry $\phi : A \rightarrow B$ such that $\phi \circ a = b \circ \phi$.

Definition 7.4. We call two primitive extensions of g -lattices

$$(A_i, a_i) \oplus (B_i, b_i) \hookrightarrow (C_i, c_i), \quad i = 1, 2$$

isomorphic if there is a commutative diagram

$$\begin{array}{ccc} (A_1, a_1) \oplus (B_1, b_1) & \longrightarrow & (C_1, c_1) \\ \downarrow & & \downarrow \\ (A_2, a_2) \oplus (B_2, b_2) & \longrightarrow & (C_2, c_2) \end{array}$$

of g -lattices.

We leave the proof of the following proposition to the reader.

Proposition 7.5. *There is a one to one correspondence between isomorphism classes of primitive extensions and the double coset*

$$\text{Aut}(A, a) \setminus \left\{ \text{Glue maps } \phi : D_A \xrightarrow{\sim} D_B \right. \\ \left. \text{satisfying } \phi \circ a = b \circ \phi \right\} / \text{Aut}(B, b)$$

where $g \cdot \phi \cdot h = g \circ \phi \circ h$ for $g \in \text{Aut}(A, a)$ and $h \in \text{Aut}(B, b)$.

Definition 7.6. A g -lattice (A, a) has few extensions if

$$\text{Aut}(A, a) \rightarrow \text{Aut}(q_A, \bar{a}) = \{g \in O(q_A) \mid g \circ a = a \circ g\}$$

is surjective.

Example 7.7. Let (A, a) be a g -lattice such that $D_A \cong \mathbb{F}_p$. Then $\text{Aut}(D_A, \bar{a}) = \{\pm \text{id}_{D_A}\}$, and we see that (A, a) has few extensions.

Lemma 7.8. *Let (A, a) be a simple $c_n(x)$ -lattice. Then $\text{Aut}(A, a) = \langle \pm a \rangle$.*

Proof. Let $h \in \text{Aut}(A, a)$. Then h is a $\mathbb{Z}[a]$ -module homomorphism, i.e. $h \in \mathbb{Z}[a]^\times \subseteq K^\times$. Since h is an isometry and (A, a) is simple, we get that

$$\text{Tr}_{\mathbb{Q}}^K(hh^\sigma x) = \text{Tr}_{\mathbb{Q}}^K(x) \quad \forall x \in K.$$

By non-degeneracy of the trace form, we get $hh^\sigma = 1$, i.e. $|h| = 1$. By Kronecker's theorem, h is a root of unity. \square

Proposition 7.9. *Let $(L_0(a), f)$ be a twist of the principal, simple $p(x)$ -lattice and $I < \mathcal{O}_K$ such that $D_{L_0(a)} \cong \mathcal{O}_K/I$. Then*

$$\text{Aut}(q_{L_0(a)}, \bar{f}) = \{[u] \in (\mathcal{O}_K/I)^\times \mid uu^\sigma \equiv 1 \pmod{I}\}.$$

Proof. Let $g \in \text{Aut}(D_{L_0(a)}, \bar{f})$. Under the usual identifications $g = [u] \in (\mathcal{O}_K/I)^\times$. Note that $(f - f^{-1})\mathcal{O}_K = \mathcal{D}_k^K$ is the relative different of K/k , and set $d = f - f^{-1}$. Then $L_0(a)^\vee = \frac{1}{ad}\mathcal{O}_K$ and $I = ad\mathcal{O}_K$ (Lemma 3.13). Since $[u]$ preserves the discriminant form, we get that $b(x, y) = b([u]x, [u]y)$ for all $x, y \in \frac{1}{ad}\mathcal{O}_K$, i.e.

$$\text{Tr}_{\mathbb{Q}}^K \left(\frac{1 - uu^\sigma}{r'(f + f^{-1})ad^2} \mathcal{O}_K \right) \subseteq \mathbb{Z}.$$

By definition of the different, this is equivalent to

$$\frac{1 - uu^\sigma}{r'(f + f^{-1})ad^2} \in \mathcal{D}_K^{-1},$$

and consequently

$$1 - uu^\sigma \in ad^2 r'(f + f^{-1}) \mathcal{D}_K^{-1}.$$

Now, the different $\mathcal{D}_K = p'(f)\mathcal{O}_K = (f - f^{-1})r'(f + f^{-1})\mathcal{O}_K$. Hence

$$1 - uu^\sigma \in ad\mathcal{O}_K = I$$

as claimed. Conversely, let $u \equiv 1 \pmod{I}$. A similar computation shows that the discriminant quadratic form $q_{L_0(a)}$ is preserved if and only if

$$1 - uu^\sigma \in add^\sigma \mathcal{O}_k = a(\mathcal{D}_k^K)^2 \cap k.$$

However, we already know $1 - uu^\sigma \in a\mathcal{D}_k^K \cap k$. By simplicity, we know that the norm $N(\mathcal{D}_k^K) = |p(1)p(-1)|$ is squarefree, and hence

$$\mathcal{D}_k^K \cap k = (\mathcal{D}_k^K)^2 \cap k.$$

□

Remark 7.10. Instead of $|p(-1)p(1)|$ being squarefree, one may assume K/k to be tamely ramified. An instance where this fails is for $\mathbb{Q}[\zeta_{2^k}]$. Here $\mathcal{D}_k^K \cap k$ is the prime ideal of \mathcal{O}_k above 2. There we need $uu^\sigma \equiv 1 \pmod{I\mathcal{D}_k^K \cap k}$.

Lemma 7.11. *All entries in Table 1 except $(24, 2^6 3^4)$ have simple glue.*

Proof. We do the calculation for $(27, 3^3)$. The other cases are similar. Set $\zeta = \zeta_{27}$. Then $I = (1 - \zeta)^3$, and $\mathcal{O}_K/I = \mathbb{Z}[\zeta]/(1 - \zeta)^3$ has 18 units. They are given by

$$u_\epsilon = \epsilon_0 + \epsilon_1(1 - \zeta) + \epsilon_2(1 - \zeta)^2, \quad (\epsilon_0 \in \{1, 2\}, \epsilon_1, \epsilon_2 \in \{0, 1, 2\})$$

We compute that

$$0 \stackrel{!}{\equiv} (1 - u_\epsilon u_\epsilon^\sigma) \equiv \epsilon_0(\epsilon_1 + \epsilon_2)(2 - \zeta - \zeta^{-1}) \pmod{(1 - \zeta)^3}.$$

We get 6 distinct solutions. However $\pm \zeta^k$ for $k \in 1, 2, 3$ are all distinct modulo $(1 - \zeta)^3$. The claim follows. □

For $n \in \mathbb{N}$ we denote by \mathcal{S}_n the symmetric group of n elements and by \mathcal{D}_n the dihedral group - the symmetry group of a regular polygon with n sides.

Lemma 7.12. *Let L be a hyperbolic lattice. Fix a chamber of the positive cone and denote by $O^+(L)/W(L) \cong \Gamma(L) \subseteq O^+(L)$ the subgroup generated by the isometries preserving the chamber. Set*

$$\phi : \Gamma(L) \rightarrow O(q_L)$$

then for $L \neq U(3) \oplus A_2$ in Table 4 ϕ is surjective. For $L = U(3) \oplus A_2$ the cokernel of ϕ is generated by $-id$. It is injective as well for all L in the table except $U(2) \oplus 2D_4$ and $U(2) \oplus D_4 \oplus E_8$ where its kernel is of order 2.

TABLE 4. Symmetry groups of a chamber

L	$\Gamma(L)$	L	$\Gamma(L)$
$U \oplus A_2$	\mathcal{S}_2	$U \oplus 2A_2$	\mathcal{D}_4
$U(3) \oplus A_2$	\mathcal{D}_4	$U(3) \oplus 2A_2$	$\mathcal{S}_6 \times \mathcal{S}_2$
$U \oplus 4A_1$	\mathcal{S}_4	$U \oplus D_4$	\mathcal{S}_3
$U(2) \oplus D_4$	\mathcal{S}_5	$U \oplus E_6$	\mathcal{S}_2
$U \oplus E_8$	\mathcal{S}_1	$U(2) \oplus 2D_4$	$\mathcal{S}_8 \times \mathcal{S}_2$
$U \oplus D_4 \oplus E_8$	\mathcal{S}_3	$U(2) \oplus D_4 \oplus E_8$	$\mathcal{S}_5 \times \mathcal{S}_2$

Proof. In all cases we can compute a fundamental root system using Vinberg's algorithm [51, §3]. An isometry preserves the chamber corresponding to the fundamental root system if and only if it preserves the fundamental root system. We get a sequence

$$0 \rightarrow O^+(L)/W(L) \cong \Gamma(L) \rightarrow \text{Sym}(\Gamma) \rightarrow 0$$

where $\text{Sym}(\Gamma)$ denotes the symmetry group of the dual graph of a fundamental root system. Since the fundamental roots form a basis of $L \otimes \mathbb{Q}$, the sequence is exact. The calculation of $\ker \phi$ is done by computer. For $L = U(2) \oplus 2D_4$ see also [26, 2.6]. \square

Remark 7.13. The observation that for $L = U(3) \oplus A_2$, $-id|_{D_L}$ generates the cokernel of ϕ gives another proof that $(54, 3^3)$ is not realized.

Proof of Theorem 7.1. Fix some pair (X, G) with (n, d) and write $G = \langle g \rangle$ for $g \in G$ such that $g^*\omega = \zeta_n\omega$. In order to prove the theorem, we have to show that $(H^2(X, \mathbb{Z}), g)$ is unique up to isomorphism as a g -lattice. We have seen that (n, d) determines $(T, g|_T)$ (and X/\cong by Thm. 5.4). By Lemma 7.11, $(T, g|_T)$ has simple glue. Hence the isomorphism class of $(H^2(X, \mathbb{Z}), g)$ is determined by the isomorphism class of $(\text{NS}, g|_{\text{NS}})$. What remains is to determine all possible isomorphism classes for $(\text{NS}, g|_{\text{NS}})$ and (n, d) fixed. This is done in the following lemmas. \square

Lemma 7.14. *For (p, p) , $p = 13, 17, 19$, $g|_{\text{NS}} = id$.*

Proof. Since the order of g on NS is strictly smaller than $n = p$ in these cases, it can only be one. \square

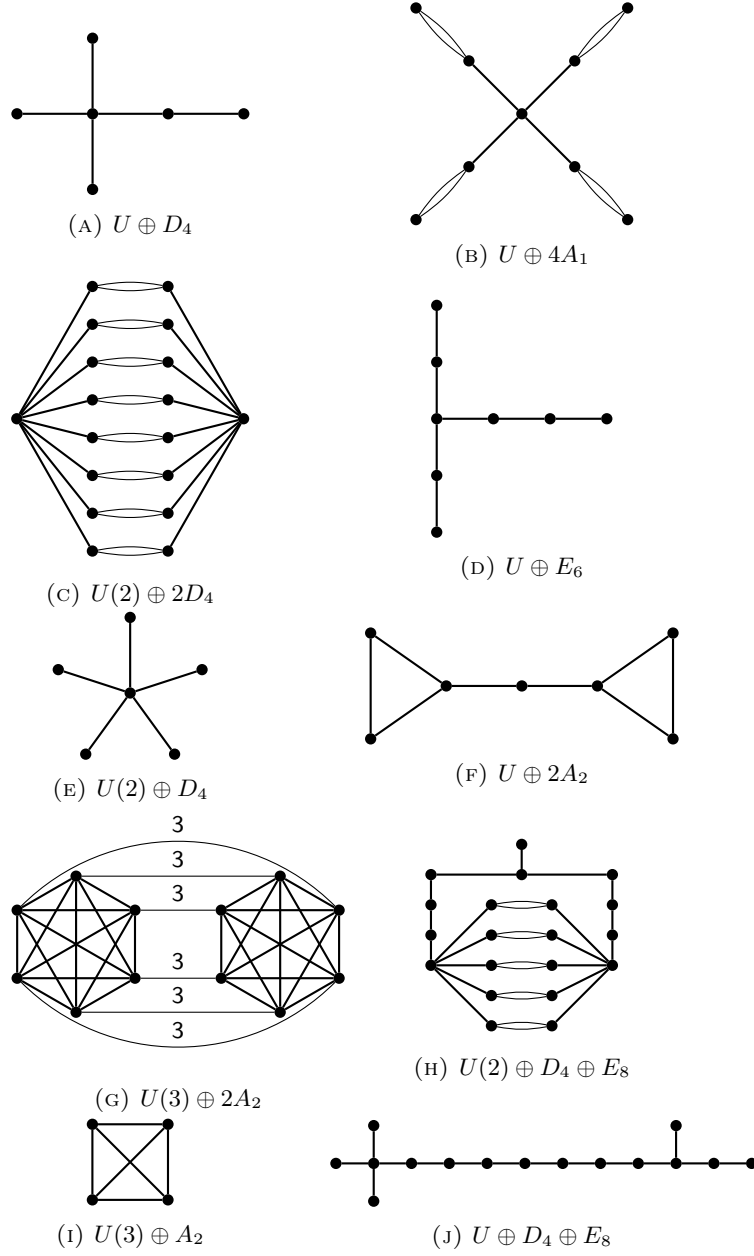


FIGURE 2. Dynkin diagrams of the fundamental root systems

Proof of Theorem 7.2. (1) By Lemma 7.12, we have for these lattices that

$$O^+(\text{NS})/W(\text{NS}) \cong O(q_{\text{NS}}).$$

Consequently, automorphisms are determined by their action on the transcendental lattice and this group is generated by $g_{(n,d)}$. (2) In this case $\phi : \Gamma(\text{NS}) \rightarrow O(q_{\text{NS}})$ has a kernel of order two and there are exactly two possibilities for $g|_{\text{NS}}$. They

differ by an element of the kernel corresponding to a symplectic automorphism of order two. \square

We note the following theorem for later use.

Theorem 7.15. [35, 3.3.14 (ii)] *Let (L, f) be a c_n -lattice of rank $m\phi(n)$.*

$$\det L \in \begin{cases} p^m \cdot (\mathbb{Q}^\times)^2, & \text{for } n = p^k, p \neq 2, \\ (\mathbb{Q}^\times)^2, & \text{else.} \end{cases}$$

Recall the notation $Ci = \ker c_i(g|H^2(X, \mathbb{Z}))$, $CiCj = \ker c_i c_j(g|H^2(X, \mathbb{Z}))$, and note that $g|_{\text{NS}}$ preserves a chamber of the positive cone if and only if

$$\ker (g^{n-1} + g^{n-2} \cdots + 1)|_{\text{NS}}$$

is root free. In this case we call g unobstructed and obstructed else.

Lemma 7.16. *For (38, 19) set*

$$R = \begin{pmatrix} -2 & -1 \\ -1 & -10 \end{pmatrix}.$$

Then $\text{NS} \cong U \oplus R$ and $g|_{\text{NS}}$ is given by the gluing of

$$C1 \cong U \oplus (-2) \text{ and } C2 \cong (-38)$$

along 2.

Proof. There are 3 cases

$$\chi_{g|_{\text{NS}}} = (x-1)^r(x+1)^{4-r}, \quad (r = 1, 2, 3).$$

Note that $C1$ is 2-elementary and $\det C1 \mid 2^m$ where $m = \min\{r, 4-r\}$ (Thm 3.8).

$r = 1$: Here $C1 = (2)$ and $C2 = (-2) \oplus R$ which is the unique even, negative definite lattice of determinant -38 .

$r = 2$: We see $\det C2 \mid 2^2 19$ and there are two such lattices - R and $R(2)$. However, R has roots, and $R(2)$ has wrong 19 glue, since the Legendre symbol $(\frac{2}{19}) = -1$.

$r = 3$: We have the single choice $C2 = (-38)$ and $C1 = U \oplus (-2)$. Indeed the gluing exists and is unique. \square

Lemma 7.17. (34, 17) *There are two pairs (X, g_1) and (X, g_2) for (34, 17). The action of $g_1|_{\text{NS}}$ is given by the gluing of*

$$C1 = U \oplus (-2) \oplus (-2) \quad \text{and} \quad C2 = - \begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}.$$

along 2.

The action of $g_2|_{\text{NS}}$ is given by the gluing of

$$C1 = (2) \oplus (-2) \oplus (-2) \quad \text{and} \quad C2 = -2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

along 2.

Proof. Here we need a little more work. Note that $\text{NS} \in II_{(1,5)}(17^{-1})$. There are the 5 cases

$$\chi_{g|\text{NS}} = (x-1)^r(x+1)^{6-r}, \quad r \in \{1, \dots, 5\}.$$

In any case $17 \mid \det C2 \mid 2^m 17$ where $m = \min\{r, 6-r\}$. The 17 part of the genus symbol of $C2$ is 17^{-1} , and moreover $2(D_{C2})_2 = 0$. Then the genus symbol looks as follows $II_{(0,6-r)}(2_*^v 17^{-1})$.

$r = 1$: Then $C1 = (2)$ and $\det C2 = -34$. Hence in order to glue above 2, $C2$ must be an element of $II_{(0,5)}(2_7 17^{-1})$ or $II_{(0,5)}(2_3^{-1} 17^{-1})$, but both genera are empty as they contradict the oddity formula.

$r = 2$: Here $C2$ is even of signature $(0, 4)$ and determinant $d = -17, -2 \cdot 17, -4 \cdot 17$. Looking at the tables in [39], we see that there are 1, 0, 7 such forms, and all of them contain roots.

$r = 3$: From the tables in [12], we extract the following. If $v = 0, \pm 2$ the respective genera are empty. If $v = 1$, there is a single genus, namely $II_{(0,3)}(2_1^1 17^{-1})$ containing two classes - both have maximum -2 . However, for $v = 3$ there are 9 negative definite ternary forms of determinant $2^3 17$. Only a single one of them has the right 2-genus symbol and no roots. It is given by

$$C2 = -2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Indeed, here $C1 = (2) \oplus (-2) \oplus (-2)$ works just fine, and as $|O(q_{C1})| = 2$ it is evident that the gluing is unique as well.

$r = 4$: Here $\det C2 \mid 2^2 17$, and we get the possibilities

$$-\begin{pmatrix} 2 & 0 \\ 0 & 34 \end{pmatrix}, -\begin{pmatrix} 4 & 2 \\ 2 & 18 \end{pmatrix}, -\begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}.$$

The first two have wrong 17 glue. We are left with the third one. It has

$$(q_R)_2 \cong (1/2) \oplus (1/2).$$

Then there is the single possibility $C1 \cong U \oplus (-2) \oplus (-2)$. Surjectivity of $O(C1) \rightarrow O(q_{C1})$, hence uniqueness of the extension is provided by Theorem 3.3.

$r = 5$: Here $C2 = (-34) \in II_{(0,1)}(2_7 17)$ has wrong 17-glue. □

For the next lemma we use the holomorphic (see [5, p.542] and [6, p.567]) and topological Lefschetz formula. We give a short account. See [48] for a similar application.

Recall that g is a purely non-symplectic automorphism of the K3 surface X with $g^* \omega = \zeta_n \omega$, where $0 \neq \omega \in H^0(X, \Omega_X^2)$. Let x be a fixed point of g . Then the local action of g at x can be linearized and diagonalized (in the holomorphic category). We call it of type (i, j) if it is of the following form

$$\begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^j \end{pmatrix}.$$

This implies that the fixed point set X^g is the disjoint union of isolated fixed points and smooth curves C_1, \dots, C_N . Set

$$a_{ij} = \frac{1}{(1 - \zeta_n^i)(1 - \zeta_n^j)} \quad \text{and} \quad b(g) = \frac{(1 + \zeta_n)(1 - g)}{(1 - \zeta_n)^2}.$$

Denote by $m_{i,j}$ the number of isolated fixed points of type (i, j) , and set $g_l = g(C_l)$ the genus of the fixed curve C_l . The topological Lefschetz formula is

$$e(X^g) = \sum_{i=0}^4 (-1)^i \text{Tr}(g^* | H^i(X, \mathbb{Z}))$$

which in our setting amounts to

$$M + \sum_{l=1}^N (2 - 2g(C_l)) = 2 + \text{Tr}(g^* | T) + \text{Tr}(g^* | \text{NS})$$

where $M = \sum_{\substack{i+j=n+1 \\ 1 \leq i \leq j \leq n}} m_{i,j}$ is the number of isolated fixed points. The holomorphic Lefschetz formula is

$$1 + \overline{\zeta_n} = \sum_{i=0}^2 (-1)^i \text{Tr}(g^* | H^i(X, \mathcal{O}_X)) = \sum_{\substack{i+j=n+1 \\ 1 \leq i \leq j \leq n}} a_{ij} m_{ij} + \sum_{l=1}^N b(g_l).$$

Lemma 7.18. *For (26, 13) the action of $g|_{\text{NS}}$ is unique and given by the gluing of*

$$C1 = U \oplus D_4 \oplus A_1 \quad \text{and} \quad C2 = - \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 10 \end{pmatrix}$$

along 2^3 .

Proof. We already know the uniqueness of (X, g^2) . One can check that g^2 has 9 isolated fixed points and a (pointwise) fixed curve of genus 1. The local types are given by

$$m_{2,12} = 3, m_{3,11} = 3, m_{4,10} = 2, m_{5,9} = 1.$$

Since $X^g \subseteq X^{g^2}$, either g fixes a curve of genus 0 and at most 9 isolated points, or g does not fix a curve and at most 11 points.

A calculation of the holomorphic Lefschetz formula yields the following possibilities: X^g fixes a curve of genus zero and 7 or 9 points. The possible local actions are

$$m_{2,25} = 4, m_{5,22} = 1, m_{11,16} = 1, m_{12,15} = 1;$$

$$m_{2,25} = 4, m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1;$$

$$m_{2,25} = 3, m_{3,24} = 2, m_{4,23} = 2, m_{5,22} = 1, m_{11,16} = 1.$$

X^g fixes 4, 5, 6 or 7 points with local contributions

$$m_{5,22} = 1, m_{7,20} = 1, m_{11,16} = 1, m_{12,15} = 1;$$

$$m_{5,22} = 1, m_{11,16} = 1, m_{12,15} = 1, m_{13,14} = 2;$$

$$m_{7,20} = 1, m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1;$$

$$m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1, m_{13,14} = 2.$$

In any case the fixed locus has Euler characteristic $e(X^g) \in \{4, 5, 6, 7\}$. Write

$$\chi_{g|_{\text{NS}}} = (x - 1)^r (x + 1)^{10-r}$$

for the characteristic polynomial of the action of g on NS. Then the topological Lefschetz formula reads

$$e(X^g) = 2 + \text{Tr}(g^*|T) + \text{Tr}(g^*|\text{NS}) = 2 + 1 + r - (10 - r) = 2r - 7,$$

and consequently $2r \in \{11, 12, 13, 14\}$, i.e., r is either 6 or 7.

We view NS as a primitive extension of $C1 \oplus C2$. Since $\text{res}(c_1, c_{26}) = 1$, we see that $13 \mid \det C2$. Further, $C1$ is 2-elementary. We conclude that $|\det C2| = 2^k 13$ where $k \leq \min\{r, 10 - r\}$.

$r = 6$: Looking at the tables in [39], we see that for $k = 0, 1, 2, 3$ all even forms of signature $(0, 4)$ and determinant $-2^k 13$ have roots. For $k = 4$ there are three forms without roots. However, none of them has 2-discriminant $(\mathbb{Z}/2\mathbb{Z})^4$.

$r = 7$: Here we use the tables of [12] to list even forms of signature $(0, 3)$ and determinant $2^k 13$.

- For $k = 0$ there is no lattice of this determinant.
- For $k = 1$ there is a single class, but it is obstructed.
- For $k = 2$ there are two genera of this determinant, but their 2-discriminant group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
- For $k = 3$ there is a single genus with right 2 discriminant and 13 glue. It is

$$II_{(0,3)}(2_1^{-3} 13^{-1})$$

and consists of the two classes

$$- \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 26 \end{pmatrix}, \quad \text{and} \quad - \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 10 \end{pmatrix} \cong C2$$

one of which, $C2$, has no roots. Then

$$C1 \cong U \oplus D_4 \oplus A_1.$$

We have to check uniqueness of the gluing. This is provided by the surjectivity of

$$O(C1) \rightarrow O(q_{C1})$$

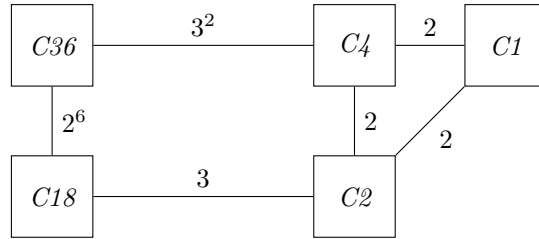
which follows from [37, 1.14.2].

□

Lemma 7.19. *For $(36, 2^6 3^2)$, the characteristic polynomial is*

$$\chi_g = c_{36} c_{18} c_4 c_2 c_1$$

and the gluings are given by



This determines the g -lattice $(\text{NS}, g|\text{NS})$ uniquely up to isomorphism.

Proof. The possible contributors to the resultant are c_9, c_{18}, c_4 and c_{12} . First the 2^6 contribution is coming from either c_9 or c_{18} dividing $\chi(g|NS)$. Then there is no room for c_{12} left. Thus the 3^2 contribution is coming from c_4 . This leaves us with

$$\chi_g = c_{36}c_{18}c_4(x \pm 1)(x - 1) \quad \text{or} \quad c_{36}c_9c_4(x \pm 1)(x - 1).$$

Since the principal $c_4(x)$ -lattice has determinant 2^2 , we have to glue it over 2^2 . This determines the characteristic polynomial to be $c_{36}c_{18}c_4c_2c_1$ or $c_{36}c_9c_4c_2c_1$. At this point we know $C36, C4 \cong (-6) \oplus (-6)$, $C18/C9 \cong E_6(2)$ and their gluings which exist by Theorem 3.7. Then

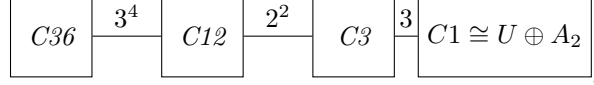
$$(q_{C1C2})_3 \cong (q_{E6(2)})_3(-1) \cong (2/3).$$

The case $C1C2 = C1 \oplus C2$ leads to $C2 = (-2)$ which is obstructed or $C2 = (-6)$ which has the wrong 3-glue. Thus we have to glue. Then $C1 \cong (4)$ and $C2 \cong (-12)$ as $C1 \cong (12)$ has wrong 3-glue. This gluing is unique since $(\mathbb{Z}/4\mathbb{Z})^\times = \{\pm 1\}$. Since $(D_{C2})_3$ can be glued to $C18$ but not to $C9$, we have

$$\chi_g = c_{36}c_{18}c_4c_2c_1.$$

The only step at which we have non-trivial freedom in the choice of glue is between $C1C2$ and $C4$. This freedom is due to the action of $\overline{g|_{C4}}$. Thus it does not affect the isomorphism class of $(C1C2C4, g_1 \oplus g_2 \oplus g_4)$ and uniqueness of (NS, g) up to isomorphism follows. \square

Lemma 7.20. *For $(36, 3^4)$ the action of $g|_{NS}$ is uniquely determined by the following gluing diagram.*



Proof.

- **Claim:** $\text{rk } C12 = 4$

The $\text{res}(c_{36}, c_4) = 3^2$ is too small. Thus $c_{12} \mid \chi_g$. Suppose $c_{12}^2 \mid \chi_g$. Then

$$\chi_g = c_{12}^2(x \pm 1)(x - 1).$$

Hence $C12$ is even of signature $(0, 8)$ and $D_{C12} \cong \mathbb{F}_3^4$. According to Magma there are two classes in this genus (one of them is $4A_2$). Both have roots. At this point we know that $C12$ is a twist of the principal c_{12} lattice.

- **Claim:** $c_3c_6, c_3^2, c_6^2 \nmid \chi_g$

In this case there is no room for c_4 and $3^4 \mid \det C12$. Now, $\text{res}(c_{12}, c_{36}c_6c_3) = 2^43^4$ and $\det C12 \in 3^4, 2^23^4, 2^43^4$. However, only for $\det C12 = 2^23^4$ there is a twist of the right signature, and we compute

$$(D_{C12})_2 \cong \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$



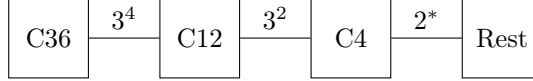
Looking at the resultant $\text{res}(c_3c_6, c_{12}c_2c_1) = 3^22^2$ and the fact that $\det C6C3$ is a square (Thm 7.15), we see that there are the two possibilities

$$\det C3C6 \in \{2^2, 2^23^2\}.$$

Either $C3C6 \cong D_4$ which is unique in its genus, or $C3C6 \in II_{(0,4)}(2^{-2}3^2)$. According to Magma this genus consists of two classes containing roots. One of them is $A_2(2) \oplus A_2$.

- **Claim:** $\det C12 = 3^4 2^2$

The resultant $\text{res}(c_{12}, c_{36}c_{3,6}c_4c_2c_1) = 2^2 3^6$, and the possible determinants of $C12$ are $3^2, 3^4, 3^6, 2^2 3^2, 2^2 3^4, 2^2 3^6$. But 3^2 has roots and for $3^4, 2^2 3^2, 2^2 3^6$ there is no twist of the right signature. This leaves us with $2^2 3^4$ or 3^6 . We show 3^6 is not possible. In this case the gluings look as follows.



In particular $(D_{C4})_3 \cong \mathbb{F}_3^2$ with a non-degenerate form must be glued to $(g^2 + 1)D_{C12}$ which is totally isotropic. This is impossible.

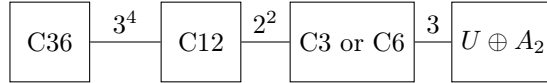
- **Claim:** $c_4 \nmid \chi_g$

Since $\det C12 = 3^4 2^2$, c_3 or c_6 must divide χ_g . This leaves us with an undetermined factor of χ_g of degree 3. Suppose c_4 divides it. Then $C4$ is principal, and by counting resultants we obtain

$$\det C4 \in \{2^2, 2^2 3^2, 2^2 3^4\}.$$

But 2^2 leads to $C4 = (-2) \oplus (-2)$ which is obstructed. If $\det C4 = 2^2 3^2$, then it must be glued over 3 to either $C36$ or $C12$. But for both the only possible glue is totally isotropic. We are left with $\det C4 = 2^2 3^4$ and then $C4$ must be glued over 3^2 with both $C36$ and $C12$. The gluing with $C12$ leads to roots, i.e. $C12C4$ is obstructed. We conclude that c_4 cannot divide χ_g .

- **Claim:**



Counting resultants yields that the determinant of $C3$ (resp. $C6$) is at most $2^2 3$, while it is at least $2^2 3$ as $C12$ needs a gluing partner. Note that $U \oplus A_2$ is the only lattice of determinant 3 and signature $(1, 3)$. A calculation shows that all gluings exist.

- **Claim:** $g|U \oplus A_2 = id$

From Lemma 7.12, we know that there are only two possibilities for $g|U \oplus A_2$. For the non-identity possibility one computes $C2 = (-6)$. However, the gluing of $C6 \cong A_2(2)$ and (-6) along 3 results in a lattice containing a root. Hence this case is obstructed.

The only case with non-trivial freedom is the gluing of $C12$ and $C3$ along 2^2 . However $\text{Aut}(D12, g|_{D12}) \rightarrow \text{Aut}((q_{D12})_2, \bar{g})$ is surjective, and hence the gluing is unique.

□

The following are the most complicated cases. The proofs are computer aided.

Lemma 7.21. *For $(21, 7^2)$ there are 3 cases distinguished by their invariant lattice:*

- (1) $U \oplus E_6$,
- (2) $U \oplus 2A_2$,
- (3) $U(3) \oplus A_2$.

Proof. Since $7^2 = \text{res}(c_{21}, c_3)$, we get that $c_3 \mid \chi_g$.

Claim: $c_7 \nmid \chi_g$.

Suppose it does. Then $\chi_g = c_{21}c_7c_3c_1^2$. The resultant $\text{res}(c_7, c_{21}c_3c_1) = 3^6 7$. But the 3^6 contribution is coming from $C21$. Hence $\det C7 = 7$ and $C7 \cong A_6$ which is a root lattice.

We distinguish cases by $\text{rk } C3$. Note that

$$\text{NS} \cong U(7) \oplus E_8 \in II_{(1,9)}(7^{-2}).$$

• **rk $C3 = 2$:**

Clearly $C3 = A_2(7) \in II_{(0,2)}(3^{-1}7^{-2})$ is root free, and we can take $C1 = U \oplus E_6$ which has simple glue by Theorem 3.3.

$$\boxed{C21} \xrightarrow{7^2} \boxed{C3} \xrightarrow{3} \boxed{C1 \cong U \oplus E_6}$$

• **rk $C3 = 4$:**

$C3 \in II_{(0,4)}(3^2 7^{-2})$ There are 3 classes in this genus:

$$A_2 \oplus A_2(7), \quad \begin{pmatrix} -6 & 3 \\ 3 & -12 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 1 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & -1 & 8 & 3 \\ 1 & 2 & 3 & 8 \end{pmatrix}$$

Obviously the first two contain roots. The third one does not, and there is an isometry

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

as well. A computation shows that this isometry represents the only conjugacy class of elements with characteristic polynomial c_3^2 . Take $C1 = U \oplus A_2 \oplus A_2$ which has simple glue by Theorem 3.3 as well.

• **rk $C3 = 6$:**

- (1) $C3 \in II_{(0,6)}(3^{-1}7^{-2})$ This genus contains 4 classes all of which contain roots.
- (2) $C3 \in II_{(0,6)}(3^3 7^{-2})$ There are 9 classes in this genus. Two of them without roots. Of these only a single one has an isometry of characteristic polynomial c_3^3 . We note that there is only a single conjugacy class.

$$C3 = \begin{pmatrix} 4 & -2 & 1 & 2 & 1 & 0 \\ -2 & 4 & -2 & -1 & -2 & 0 \\ 1 & -2 & 4 & 2 & 2 & -1 \\ 2 & -1 & 2 & 4 & 1 & -2 \\ 1 & -2 & 2 & 1 & 6 & 2 \\ 0 & 0 & -1 & -2 & 2 & 6 \end{pmatrix}, \quad g|C3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We can take $C1 = U(3) \oplus A_2$. We have seen the surjectivity of $O(C1) \rightarrow O(q_{C1})$ already in the proof of Lemma 6.4. Hence $C1$ has simple glue and the construction is unique.

- **rk $C3 = 8$:**

In this case $\det C3$ is a square and dividing $3^2 7^2$. Hence there are two possibilities for the genus of $C3$.

- (1) $C3 \in II_{(0,8)}(7^{-2})$ There are three classes in this genus. All of them contain roots.
- (2) $C3 \in II_{(0,8)}(3^{-2} 7^{-2})$ contains the single class $A_2(7) \oplus E_6$ which is obstructed.

□

Lemma 7.22. *For $(42, 7^2)$ there are exactly two actions of $g|NS$ distinguished by*

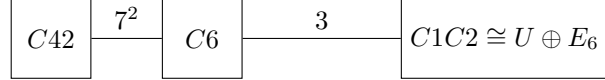
- (1) $C1C2 \cong U \oplus E_6$, $\chi(g|NS) = c_6 c_2^2 c_1^6$.
- (2) $C1C2 \cong U \oplus 2A_2$, $\chi(g|NS) = c_6 c_3 c_2 c_1^5$.

Proof. We distinguish along the three cases of $(21, 7^2)$. Recall that $C3C6$ is glued along 7^2 with $C42$. Hence

$$\chi(g|(D_{C3C6})_7) = \chi(g|D_{C42}) = x^2 - x + 1.$$

- **rk $C3C6 = 2$:**

In this case we have the following glue diagram,



It remains to determine $g|U \oplus E_6$. This is the content of Lemma 7.12. Uniqueness of the glue is evident.

- **rk $C3C6 = 4$:**

There are two conjugacy classes $[f_1], [f_2]$ of $O(C3C6)$ with the right characteristic polynomials:

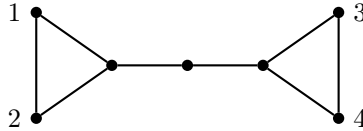
$$(1) f_1 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \chi(f_1) = c_3 c_6;$$

$$(2) f_2 = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \chi(f_2) = c_6^2.$$

On the discriminant form we get

- (1) $(q, \bar{f}_1) \cong [(2/3) \oplus (2/3), id \oplus -id]$,
- (2) $(q, \bar{f}_2) \cong [(2/3) \oplus (2/3), -id \oplus -id]$.

Here $C1C2 \cong U \oplus 2A_2$ and Lemma 7.12 yields the fundamental root system:



Its symmetry group $O^+(C1C2)/W(C1C2) \cong \mathcal{D}_4 \cong \langle (12), (13)(24) \rangle$ is of order 8. It has 5 conjugacy classes. One of them is of order 4 which is too high. The other ones and their action on q_{C1C2} are represented by

- (1) $()$, $[(1/3) \oplus (1/3), id \oplus id]$;

- (2) (34), $[(1/3) \oplus (1/3), id \oplus -id]$;
- (3) (12)(34), $[(1/3) \oplus (1/3), -id \oplus -id]$;
- (4) (13)(24), $[(2/3) \oplus (2/3), id \oplus -id]$.

Comparing the actions, there are two combinations for a gluing.

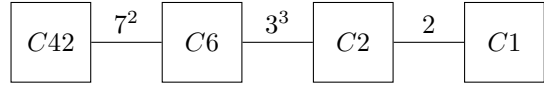
- f_2 and (12)(34) is obstructed since new roots appear.
- f_1 and (34) works.

Since $Aut(q_{C_1C_2}, \bar{g}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is generated by the images of $-id|C_1C_2$ and (34), it has simple glue.

• **rk $C_3C_6 = 6$:**

In this case there are two possible isometries with the right characteristic polynomials. Namely,

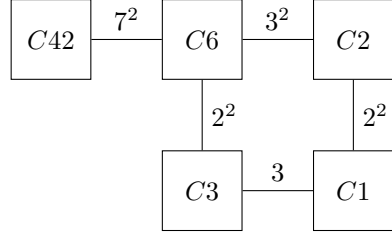
- (1) with characteristic polynomial c_6^3 , In this case we have the following gluings



and there $C_2 \in II_{(0,3)}(2_+^{\pm} 3^{-3})$. By the tables in [12] there is no such lattice. (Alternatively check [14, Chap. 15, (31-35)].)

- (2) With characteristic polynomial $c_3c_6^2$.

- (a) rk $C_2 = 2$ Then $\det C_2 \mid 3^2 2^2$, and the only possibility is $C_2 = (-6) \oplus (-6)$. Then the gluings look as follows



Then $C_1 = (6) \oplus (-2)$ or $(2) \oplus (-6)$, but only the first one glues along 2 with C_2 as well. Now, $C_3 \cong A_2(2)$ must be glued to (6) along 3. This is impossible.

- (b) rk $C_2 = 3$ and $C_1 = (6)$. Hence $\det C_2 = -2 \cdot 3^2$, and consequently $C_2 = A_2 \oplus (-2)$ is obstructed.

□

Lemma 7.23. *Affine Weierstraß models for $X_{(21,7^2)}$ and the automorphisms of order 21 and 42 corresponding to the cases (1),(2) in Lemmas 7.21 and 7.21 are given below. For case (3) there is a singular projective model.*

- (1) $y^2 = x^3 + t^4(t^7 + 1), \quad (x, y, t) \mapsto (\zeta_3 \zeta_7^6 x, \pm \zeta_7^2 y, \zeta_7 t);$
- (2) $y^2 = x^3 + t^3(t^7 + 1), \quad (x, y, t) \mapsto (\zeta_3 \zeta_7^3 x, \pm \zeta_7 y, \zeta_7 t);$
- (3) $x_0^3 x_1 + x_1^3 x_2 + x_0 x_2^3 - x_0 x_3^3, \quad (x_0, x_1, x_2, x_3) \mapsto (\zeta_7 x_0, \zeta_7 x_1, x_2, \zeta_3 x_3).$

Proof. We identify the three cases by computing the fixed lattice of f^{14} .

- (1) There is an E_6 fiber at $t = 0$.
- (2) There is a fiber of type I_0^* at $t = 0$ and a fiber of type IV at $t = \infty$. Now g^{14} fixes exactly one isolated point in each fiber and a curve of genus 3. This leads

- to the fixed lattice $U \oplus A_2 \oplus A_2$ of g^{14} . Here U consists of the zero section and the class of a fiber, one A_2 is in the IV fiber and the other one in the I_0^* fiber, namely the component of multiplicity 2 and the one meeting the zero section.
- (3) There are 3 singularities of type A_2 located at the points $(0 : 0 : 1 : \zeta_3^k)$. The fixed points of g^{14} are a smooth curve of genus 3 at $x_3 = 0$ and the isolated point $(0 : 0 : 0 : 1)$. Hence g^{14} has invariant lattice $U(3) \oplus A_2$. We see that $c_3^3 \mid \chi_{g|NS}$.

□

Remark 7.24. In general it is a hard problem to find equations for a K3 surface with a given Hodge structure. In practice equations are found by a mixture of theoretical knowledge, computer algebra, heuristics and intuition. In other words by an 'educated guess'. Below we give some heuristics.

Let (X, g) be a pair of a complex K3 surface and a finite automorphism. We want to find a (possibly singular) birational model of (X, g) . By definition $NS(X)^g$ is primitive. Since g preserves an ample class, this defines a (pseudo-)ample $C1$ -polarization on X . Let $D \in NS(X)^g$ be a (nef) divisor. Then g acts linearly on $H^0(X, D)$ and since it is of finite order (in characteristic 0), we can diagonalize this action. A relatively simple case is when we can choose some $U \subseteq C1 \subseteq L \cong NS$. This induces an equivariant elliptic fibration with reducible singular fibers given by the roots of $U^\perp \subseteq NS$. Since Weierstraß equations are quite accessible, it is often possible from this point to guess equations.

If there is no $U \subseteq NS^g$, we instead choose $D \in NS$ with $0 < D^2$ small. We may assume D effective, hence pseudo-ample. Since $D^\perp \subseteq NS$ is negative definite it is easy to compute its roots (and the action of g on them) which correspond to ADE singularities. At this point further considerations depend on the geometric situation. For example one can compute the fixed locus of g^k and try to specialize known families.

8. (NON-)SYMPLECTIC AUTOMORPHISMS OF ORDER 5

We want to consider K3 surfaces admitting a symplectic as well as a non-symplectic automorphism of order five. In this section we collect the necessary background material.

Theorem 8.1. [36] *Let G be a finite abelian group acting symplectically on a complex K3 surface X . Then the action of G on the K3-lattice is unique up to isometries of $H^2(X, \mathbb{Z})$. Hence the isometry class of $\Omega_G := (H^2(X, \mathbb{Z})^G)^\perp$ is determined by G . Conversely Ω_G is primitively embedded in $NS(X)$ if and only if G acts as a group of symplectic automorphisms on X .*

We need only the following case for our purposes.

Proposition 8.2. [36, Prop. 10.1] *If X is a K3 surface with a symplectic automorphism σ of order 5, then the invariant lattice $H^2(X, \mathbb{Z})^\sigma$ is isomorphic to $U \oplus 2U(5)$.*

Theorem 8.3. [4] *Let X be a K3 surface with a non-symplectic automorphism τ of order 5 such that τ fixes a curve of genus g and additional k curves of genus 0. Then this data is as in Table 5 and all cases occur. The number of isolated fixed points and their local type is given by n_1, n_2 .*

TABLE 5. Non-symplectic automorphisms of order 5

n_1	n_2	g	k	$(H^2(X, \mathbb{Z})^\sigma)^\perp$	$H^2(X, \mathbb{Z})^\sigma$
1	0	2	0	$H_5 \oplus U \oplus E_8 \oplus E_8$	H_5
3	1	1	0	$H_5 \oplus U \oplus E_8 \oplus A_4$	$H_5 \oplus A_4$
3	1	—	—	$H_5 \oplus U(5) \oplus E_8 \oplus A_4$	$H_5 \oplus A_4^*(5)$
5	2	1	1	$U \oplus H_5 \oplus E_8$	$H_5 \oplus E_8$
5	2	0	0	$U \oplus H_5 \oplus A_4^2$	$H_5 \oplus A_4^2$
7	3	0	1	$U \oplus H_5 \oplus A_4$	$H_5 \oplus A_4 \oplus E_8$
9	4	0	2	$U \oplus H_5$	$H_5 \oplus E_8 \oplus E_8$

Let $\tau \in O(L_{K3})$ be an isometry of prime order p with hyperbolic invariant lattice $S(\tau) = \ker(\tau - id)$ and $[\tau]$ its conjugacy class. A $[\tau]$ -polarized K3 surface is a pair (X, ρ) consisting of a K3 surface X and a non-symplectic automorphism ρ such that

$$\rho^*(\omega_X) = \zeta_p \omega_X \quad \rho^* = \phi \circ \tau \circ \phi^{-1}$$

for some marking $\phi : L_{K3} \rightarrow H^2(X, \mathbb{Z})$. We say two $[\tau]$ -polarized K3 surfaces $(X, \rho), (X', \rho')$ are isomorphic if there is an isomorphism $f : X \rightarrow X'$ with $f^{-1} \circ \rho' \circ f = \rho$. As usual set $T(\tau) = S(\tau)^\perp$ and let

$$V^\tau := \{x \in L_{K3} \otimes \mathbb{C} \mid \tau(x) = \zeta_p x\} \subseteq T(\tau) \otimes \mathbb{C}$$

be a complex eigenspace of τ . We set

$$D^\tau = \{\omega \in \mathbb{P}(V^\tau) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}$$

and

$$\Delta^\tau = \bigcup_{d \in T(\tau), d^2 = -2} D^\tau \cap d^\perp.$$

With $\Gamma^\tau = \{\gamma \in O(L_{K3}) \mid \gamma \circ \tau = \tau \circ \gamma\}$ we get the following theorem.

Theorem 8.4. [4] *The orbit space $\mathcal{M}^\tau := \Gamma^\tau \backslash (D^\tau \setminus \Delta^\tau)$ parametrizes isomorphism classes of $[\tau]$ -polarized K3 surfaces. Two pairs $(X, \rho), (X', \rho')$ of K3 surfaces with non-symplectic automorphism of prime order are polarized by the same $\rho \in O(L_{K3})$ if and only if $S(\rho) \cong S(\rho')$.*

8.1. Simultaneous symplectic and non-symplectic actions of order 5. We saw that K3 surfaces with a non-symplectic automorphism τ and transcendental lattice T of rank 4 are determined by the $c_5(x)$ -isometry class of (T, τ) . In this section we ask which $c_5(x)$ -lattices arise in this fashion from K3 surfaces and which of them admit a symplectic automorphism of order 5. Similar methods have recently been applied in [11]. There the authors prove the existence of a non-symplectic automorphism of order 23 on a holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of two points on a K3 surface.

Lemma 8.5. *A complex K3 surface X admits a symplectic automorphism of order 5 if and only if the transcendental lattice embeds primitively into*

$$T \hookrightarrow U \oplus 2U(5).$$

Proof. Proposition 8.2 provides the only if part. Now assume that

$$T \hookrightarrow U \oplus 2U(5).$$

Since T has signature $(2, 22 - \rho)$, this implies $l(D_T) \leq \text{rk } T \leq 5$. By Theorem 3.2 $T \hookrightarrow L_{K3}$ is unique and so is the orthogonal complement NS of T . It contains $\Omega_{\mathbb{Z}/5\mathbb{Z}}$. Now apply Theorem 8.1. \square

8.2. The commutative case.

Proposition 8.6. *The pair (S, G) where S is a complex K3 surface and $G = (\mathbb{Z}/5\mathbb{Z})^2$ is unique up to isomorphism. It is given as the minimal resolution of the double cover of the projective plane branched over the sextic curve given by*

$$x_0(x_0^5 + x_1^5 + x_2^5)$$

with diagonal G action. The transcendental and Néron-Severi lattice of S are isometric to

$$\text{NS} \cong H_5 \oplus 2A_4 \oplus E_8, \quad T \cong H_5 \oplus U(5).$$

Proof. Let X be a (complex) K3 surface with a faithful $G = (\mathbb{Z}/5\mathbb{Z})^2$ action. Then G/G_s is cyclic [36, Thm 3.1] where $G_s = \ker G \rightarrow O(H^{2,0}(X))$. Since G is not symplectic by the classification of (abelian) symplectic actions in [36, Thm. 4.5], $G_s \neq G$. This leaves us with $G_s \cong \mathbb{Z}/5\mathbb{Z}$. Then $G = \langle \sigma, \tau \rangle$ with a symplectic automorphism σ and τ a non-symplectic automorphism such that $\sigma \circ \tau = \tau \circ \sigma$. First we show that T has rank 4 and the action of τ on T^\vee/T is of the form $\mathcal{O}_K/(x-1)^3$. Then Theorem 5.4 provides the uniqueness of X .

The transcendental lattice T embeds in the invariant lattice of σ ,

$$i : T \hookrightarrow U \oplus 2U(5) \cong H^2(X, \mathbb{Z})^\sigma.$$

In particular $\text{rk } T \leq 5$. As X admits the non-symplectic automorphism τ of order 5. Thus $\text{rk } T \equiv 0 \pmod{4}$. This leaves us with $\text{rk } T = 4$, i.e. the Picard number of X is 18. Since τ and σ commute, τ acts on both sides of the embedding i . This observation is our starting point. Denote by $R := i(T)^\perp$ then

$$T \oplus R \hookrightarrow U \oplus 2U(5)$$

is a primitive extension compatible with the action of τ . It corresponds to a glue map ϕ

$$D_T \supseteq G_T \xrightarrow[\phi]{\sim} G_R \subseteq D_R$$

with $\phi \circ \tau = \tau \circ \phi$ such that $q_T(x) = -q_R(\phi(x))$. By definition τ acts with order 5 on T . The pair $(T, \tau|_T)$ is a $c_5(x)$ -lattice. Since $c_5(x)$ is a simple reciprocal polynomial, this pair is isomorphic to a twist $L_0(a)$ of the principal lattice

$$(T, \tau) \cong (L_0(a), f).$$

Our next goal is to determine the prime factorization of a . The rank of R is too small for an action of order 5, so there τ restricts to the identity. The resultant equals $\text{res}(x^4 + x^3 + x^2 + x + 1, x - 1) = c_5(1) = 5$. By Theorem ?? this forces $G_T \cong G_R$ to be 5-groups, i.e., gluing can occur only over 5. In view of $U \oplus 2U(5)$ being 5-elementary we know that D_T and D_R are 5-groups as well. Since D_T is a 5-group, it suffices to consider twists above 5. There is only a single prime ideal

$(x-1)\mathcal{O}_K$ over 5. Hence up to units we may only twist by associates t of $(x-1)^2$. Recall from Lemma 3.13 that

$$D(L_0(t^k)) \cong \mathcal{O}_K/(x-1)^{2k+1}.$$

Since the action of τ preserves G_T , the glue G_T is actually isomorphic to an ideal in $\mathcal{O}_K/(x-1)^{2k+1}\mathcal{O}_K$. These are of the form $(x-1)^h\mathcal{O}_K/(x-1)^{2k+1}$.

$$(2) \quad G_T \cong \mathcal{O}_K/(x-1)^{2k+1-h}\mathcal{O}_K$$

$$(3) \quad D_T/G_T \cong \mathcal{O}_K/(x-1)^h\mathcal{O}_K$$

The action of τ on $G_T \cong G_R$ is the identity. It is given by multiplying by x . This means that

$$x \equiv 1 \pmod{(x-1)^{2k+1-h}}$$

which is the case for $2k+1-h \in \{0, 1\}$. Since the gluing results in a 5-elementary lattice, D_T/G_T (and D_R/G_R) are \mathbb{F}_5 -vector spaces. We get $h \leq 4$ and then $k \in \{0, 1, 2\}$. We want to show $k = 1$.

First suppose $k = 0$, then D_T has length 1 and D_R at most length 2. However, $D_{U \oplus 2U(5)} \cong \mathbb{F}_5^4$ has length 4 and is a sub-quotient of $D_T \oplus D_R$. This is impossible. Now suppose $k = 2$, then $D(L_0(t^2)) \cong \mathbb{F}_5^3 \oplus \mathbb{Z}/25\mathbb{Z}$ which is not a vector space. Hence we have to glue, that is, $h = 4$ and $G_T = 5D_T \cong \mathbb{F}_5$. Solving

$$(4) \quad |D_T/G_T||D_R/G_R| = |D_{U \oplus 2U(5)}| = 5^4$$

we get $|D_R| = 5$. Thus we arrive at a glue map $5D_T = G_T \cong G_R = D_R$, but the discriminant form on G_T is 0 while on D_R it is non-degenerate - a contradiction. We are left with $k = 1$ and $D_T \cong \mathcal{O}_K/(x-1)^3 \cong \mathbb{F}_5^3$.

The uniqueness of a symplectic action is well known, and we have computed the conjugacy class of $\tau|(H^2(X, \mathbb{Z})^\sigma)$. Set $\Omega = (H^2(X, \mathbb{Z})^\sigma)^\perp$ and recall $\Omega \cong \Omega_{\mathbb{Z}/5\mathbb{Z}}$. From the gluing we know (the conjugacy class of) $\overline{\tau\alpha} \in O(q_\Omega)$. Hence the action of $\tau|\Omega$ is unique up to the kernel of $O(\Omega) \rightarrow O(q_\Omega)$ which is generated by σ . Finally, the few extension property of $(T, \tau|T)$ provides us with the uniqueness of τ (up to multiplication by σ) and hence G . \square

In order to extend this result to positive characteristic we recall some results and definitions concerning the lifting of an automorphism. For details, we refer to [22]. Let k be an algebraically closed field of positive characteristic p and X/k a K3 surface. We get the canonical surjection

$$\pi : H_{cris}^2(X/W) \rightarrow H_{cris}^2(X/W)/pH_{cris}^2(X/W) \cong H_{dR}^2(X/k).$$

Attached to $H_{dR}^2(X/k)$ is the Hodge filtration $F^i H_{dR}^2(X/k)$, $0 \leq i \leq 3$ where $F^2 H_{dR}^2(X/k) \cong H^0(X, \Omega_X^2)$. Any isotropic line $M \subseteq H_{cris}^2(X/W)$ with $\pi(M) = F^2 H_{dR}^2(X/k)$ corresponds to a formal lift \mathcal{X} of X/k . It is algebraic if and only if there is an ample line bundle L with $c_1(L) \in M^\perp$. An automorphism of X/k lifts to \mathcal{X} (and its algebraization) iff it preserves M .

Lemma 8.7. *Let k be an algebraically closed field of positive characteristic $p \neq 2$. Let X/k be a K3 surface, σ a symplectic and τ a non-symplectic automorphism of order five. If σ and τ commute, then the triple (X, σ, τ) lifts to characteristic zero.*

Proof. Since there is no non-symplectic automorphism of order 5 in characteristic 5, we can assume $p \neq 5$. By the discussion above we have to find an isotropic rank one submodule $M \subseteq H_{cris}^2(X/W)$ and an ample line bundle L with $c_1(L) \in M^\perp$ such that $\pi(M) = F^2 H_{dR}^2(X, k)$ and $\sigma(M) = M, \tau(M) = M$. We imitate the reasoning of [22, 3.7].

Assume that X is of finite height. As in (the proof of) [22, 3.7] we can lift (X, τ) with M inside $H^2(X/W)_{[1+1/h]} \subseteq T_{cris} = NS^\perp \subseteq H_{cris}^2(X/W)$. Since σ is symplectic, it acts as identity on T_{cris} (cf. [22, 3.5]). Hence, it trivially preserves M and σ lifts together with (X, τ) (to an algebraic K3 surface).

Now assume that X is supersingular. Set $H := H_{cris}^2(X/W)$ and let $\zeta \in W$ be a 5th root of unity such that τ acts on $F^2 H_{dR}^2(X, k) \cong H^0(X, \Omega_X^2)$ by $\bar{\zeta} = \zeta + pW$. Recall that $p \neq 5$ and hence, by Hensel's lemma, $t^5 - 1 \in W[t]$ splits. In particular the action of σ and τ on H is semisimple. The simultaneous eigenspaces

$$E_{k,l} = \ker(\sigma - \zeta^k I) \cap \ker(\tau - \zeta^l I), \quad 0 \leq k, l \leq 4$$

induce decompositions

$$H \cong \bigoplus_{k,l=1}^{22} E_{k,l} \quad \text{and} \quad H/pH \cong \bigoplus_{k,l=1}^{22} E_{k,l}/pE_{k,l}.$$

Note that $E_{k,l}/pE_{k,l}$ is the simultaneous eigenspace corresponding to $\bar{\zeta}^k$ and $\bar{\zeta}^l$. Take any $0 \neq v \in F^2 H_{dR}^2(X, k) \subseteq E_{0,1}/pE_{0,1}$ and lift it to some $m \in E_{0,1}$. Set $M = Wm$. By construction $\pi(M) = F^2 H_{dR}^2(X, k)$ and M is preserved by both σ and τ . Any eigenvector of τ with eigenvalue ζ is isotropic ($\zeta^2 \neq 1$). We can find a τ^* invariant ample line bundle. It is orthogonal to M . This completes the proof by showing that the lift induced by M is algebraic. \square

By lifting the triple (X, σ, τ) we can reduce to the complex case and get the following proposition.

Proposition 8.8. *Let k be an algebraically closed field of odd or zero characteristic not 5. There is a unique K3 surface S/k admitting a commuting pair of a symplectic and a non-symplectic automorphism of order 5. In characteristic five no such surface exists.*

Example 8.9. [16, §6] The general unitary group $GU(3, \mathbb{F}_{5^2})$ acts on the Hermitian form $x_0^6 + x_1^6 + x_2^6$ over \mathbb{F}_{5^2} . Then $\text{PSU}(3, \mathbb{F}_{5^2})$ acts symplectically on the double cover of \mathbb{P}^2 branched over $x_0^6 + x_1^6 + x_2^6 = 0$. It is a supersingular K3 surface of Artin invariant $\sigma = 1$. We note that $(\mathbb{Z}/5\mathbb{Z})^2 \hookrightarrow \text{PSU}(3, \mathbb{F}_{5^2})$.

8.3. The non-commutative case. In this section we will prove the existence of an infinite number of K3 surfaces with a non-symplectic and a symplectic automorphism of order 5. We present them as a sequence of $[\rho]$ -polarized K3 surfaces. So essentially we are constructing their Hodge structures. Then surjectivity of the period map and the Torelli-Theorem provide their existence.

Proposition 8.10. *Let X/\mathbb{C} be a K3 surface with a non-symplectic automorphism τ and a symplectic automorphism σ both of order 5. Then*

- (1) *there is a primitive embedding $i : T \hookrightarrow U \oplus 2U(5)$,*
- (2) *$(T, \tau) \hookrightarrow (N, f)$ where N is as in Table 5, f is of order 5, its characteristic polynomial is a perfect power of $c_5(x)$ and f acts as identity on N^\vee/N ,*
- (3) *the orthogonal complement C of T in N does not contain any roots.*

Conversely these conditions are sufficient for a $c_5(x)$ -lattice to arise as (T, τ) from a K3 surface.

Proof. The first condition is Lemma 8.5. To see the necessity of the second condition note that $N = (H^2(X, \mathbb{Z})^\tau)^\perp =: T(\tau)$ and $S(\tau) := H^2(X, \mathbb{Z})^\tau$. Since the K3-lattice is unimodular, we get an isomorphism $T(\tau)^\vee/T(\tau) \cong S(\tau)^\vee/S(\tau)$ compatible with the action of τ on both sides. Since τ is the identity on the right side, it is the identity on the left side as well. For the third condition, note that the orthogonal complement of T in $T(\tau)$ lies in NS . It can be shown with Riemann-Roch, that if $x \in \text{NS}$ is a root, then x or $-x$ is effective. Suppose x is. Let h be an ample class. If $x \in N \cap \text{NS}$, then $0 < h \cdot (x + \tau(x) + \tau^2(x) + \tau^3(x) + \tau^4(x)) = h \cdot 0 = 0$. Thus these roots are an obstruction for τ to preserve the effective cone in NS . Hence they do not exist.

Let us turn to the sufficiency: By (2) we can extend τ to an isometry f of $N =: T(f)$ which we can then glue to the identity on the matching $S(f)$ to obtain an isometry f on the K3-lattice. We realize X as an $[f]$ -polarized K3 surface. After replacing f by f^n and τ by τ^n , we can assume that $(\omega, \overline{\omega}) > 0$ for any non-zero $\omega \in \eta := \ker(\tau - \zeta_5 \text{id}) \subseteq T \otimes \mathbb{C}$. This eigenspace η is our candidate period in N^f . Once we show that $\eta \notin \Delta^f$ (as defined in Sect. 8.1, we can apply Theorem 8.4. Assume $\exists d \in T(f)$ with $d^2 = -2$ and $(d, \omega) = 0$. Then $d \in \eta^\perp \cap T(f) = C$, but C has no roots. Hence, such d do not exist. We get the existence of an $[f]$ -polarized K3 surface X with period η .

In particular X has a non-symplectic automorphism of order 5 and transcendental lattice isometric to T . Then (1) and Lemma 8.5 imply that X has a symplectic automorphism of order 5 as well. □

We start our search for different K3 surfaces with both a symplectic and a non-symplectic automorphism of order 5 by analyzing (1). Set $C = i(T)^\perp$. Then $l(D(T)_p) = l(D(C)_p) \leq 2$ for $p \neq 5$. Hence we cannot twist by inert primes - these result in length 4. We can only twist by the prime above 5 or by primes above $p \equiv 1, 4 \pmod{5}$.

Lemma 8.11. *Let r_1, \dots, r_n be primes in \mathcal{O}_k above the distinct primes $p_1, \dots, p_n \equiv 1 \pmod{5}$ and s be the prime over 5. Then for $r = \prod_i r_i$*

$$L_0(sr) \hookrightarrow U \oplus 2U(5)$$

primitively given that $L_0(sr)$ has signature $(2, 2)$.

Proof. A different perspective to primitive embeddings is a primitive extension. We glue $L_0(sr)$ and $H_5(\prod_i p_i)$ to obtain $U \oplus 2U(5)$. Since $p_i \equiv 1 \pmod{5}$, the prime $r_i \in \mathcal{O}_k$ splits in \mathcal{O}_K as $r_i = r_{i1}r_{i2}$.

$$D(L_0(sr))_{p_i} \cong \mathcal{O}_K/r_i \cong \mathcal{O}_K/r_{i1} \times \mathcal{O}_K/r_{i2} \cong \mathbb{F}_{p_i}^2$$

And the form on $D(L_0(sr))_{p_i}$ in a basis of eigenvectors can be normalized to

$$q_{p_i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular q_{p_i} has determinant $-1 \in \mathbb{F}_{p_i}^\times / \mathbb{F}_{p_i}^{\times 2}$. Since the dimension is even, $q_{p_i} \cong q_{p_i}(-1)$. So for a glue map to exist it is enough to show that the discriminant form on $D(H_5(p))_{p_i}$ is isomorphic to q_{p_i} . It can be computed directly:

$$\det 5 \prod_i p_i \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = -5^3 \prod_i p_i^2.$$

Its square class is given by the Legendre symbol

$$\left(\frac{-5^3}{p_i} \right) = \left(\frac{-1}{p_i} \right) \left(\frac{p_i}{5} \right) = \left(\frac{-1}{p_i} \right)$$

as desired. \square

So for condition (1) we have a nice list of examples. It remains to check conditions (2) and (3). Set $T := L_0(sr)$. We are searching for a gluing of $c_5(x)$ -lattices

$$(T, f_T) \oplus (C, f_C) \hookrightarrow (N, f)$$

where N is a $(x-1)$ -elementary c_5 -lattice as in Table 5. As a first try we can take $N \cong U \oplus H_5$. Then $C = 0$ and $T \cong N$ which is not the case. As a second try take $N \cong U \oplus H_5 \oplus A_4$. We will see that it does not work and develop along the way the methods to handle the third try successfully.

Now C is of rank 4. So by Theorem 3.11 it is a twist of the principal $c_5(x)$ -lattice as well. Since N is 5-elementary, the $p \neq 5$ -parts of the discriminant groups of T and C are isomorphic. For the 5-glue we use that N is $(x-1)$ elementary. In particular $(D_T/G_T)_5 \cong \mathcal{O}_K/(x-1)$ which implies that $(D_C/G)_5 \cong \mathcal{O}/(x-1)$ as well. We end up with a primitive extension

$$L_0(sr) \oplus L_0(\epsilon sr) \hookrightarrow U \oplus H_5 \oplus A_4$$

where $\epsilon \in \mathcal{O}_k^\times$ is chosen such that $C := L_0(\epsilon sr)$ has signature $(0, 4)$. Since we have to glue over 5, we need some more knowledge of how to glue $c_5(x)$ -lattices.

Lemma 8.12. *Let q be a non-degenerate quadratic form on $\mathbb{F}_5[X]/(X-1)^3$ where multiplication by X is an isometry. Then q can be normalized as follows:*

$$\begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

In the first case $\det q$ is a square and the second case not. In any case $a \in \mathbb{F}_5^\times$ and the basis is given by $u \cdot 1, u \cdot (X-1), u \cdot (X-1)^2$ for some unit $u \in \mathbb{F}_5^\times$.

Proof. We start in the basis $1, X, X^2$ of $\mathbb{F}_5[X]/(X-1)^3$. By invariance under multiplication by X the Gram-matrix of q is of the form

$$\begin{pmatrix} a & b & 4b-3a \\ b & a & b \\ 4b-3a & b & a \end{pmatrix}.$$

It has determinant $8(b-a)^3$. We can change the basis to $1, X-1, (X-1)^2$. In this basis the Gram-matrix is given by

$$\begin{pmatrix} a & b-a & 2(b-a) \\ b-a & -2(b-a) & 0 \\ 2(b-a) & 0 & 0 \end{pmatrix}$$

After multiplying the basis by an element $u \in \mathbb{F}_5^\times$ we can assume that $(b-a) \in \{1, 2\}$. Finally, by replacing 1 by $1 + u(x-1)^2$ for some $u \in \mathbb{F}_5$, we get $a = 0$. \square

Lemma 8.13. *Let q_1 and q_2 be isomorphic quadratic forms over $\mathcal{O}_K/(x-1)^3$ invariant under multiplication by x . Let $G_1 = G_2 = (x-1)\mathcal{O}_K/(x-1)^3$. Then we can find an \mathcal{O}_K -module isomorphism $\phi : G_1 \rightarrow G_2$ with $q_1(x) = -q_2(\phi(x))$ and graph Γ such that $\Gamma^\perp/\Gamma \cong (\mathcal{O}_K/(x-1))^2$.*

For $\tilde{G}_i = (x-1)^2\mathcal{O}_K/(x-1)^3$ we can find a glue map with

$$\Gamma^\perp/\Gamma \cong \mathcal{O}_K/(x-1)^3 \oplus \mathcal{O}_K/(x-1).$$

This sum can be chosen orthogonal. The square class of the $\mathcal{O}_K/(x-1)^3$ -part is independent of choices and different from that of q_1 .

Proof. Assume $\det q_1$ a square mod 5. First we normalize the forms as in Lemma 8.12 (recall that -1 is a square mod 5). That is, we can find $v \in \mathbb{F}_5[X]/(x-1)^3$ such that the q_i are given by the following matrices

$$q_1 = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 0 & -2 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in the bases $e_i = (x-1)^i v$ for q_1 and $b_i = 2(x-1)^i v$, for q_2 , $i \in \{0, 1, 2\}$.

Then we define ϕ by $\phi(e_1) = b_1$ and $\phi(e_2) = b_2$. By construction this reverses the signs and is an \mathcal{O}_K -module isomorphism. It remains to compute Γ^\perp/Γ where

$$\Gamma = \langle e_1 + b_1, e_2 + b_2 \rangle.$$

Thus

$$\Gamma^\perp = \langle e_2, b_2, e_0 + b_0, e_1 + b_1 \rangle.$$

By definition multiplication by $(x-1)$ raises the index of the e_i, b_i by one. Hence we get the desired module structure of Γ^\perp/Γ . For \tilde{G}_i , take $\phi(e_2) = b_2$ and do the computation. The proofs are the same for $\det q_i$ a non-square. \square

Let us return to the hoped for primitive extension

$$L_0(sr) \oplus L_0(\epsilon sr) \hookrightarrow U \oplus H_5 \oplus A_4.$$

In order to glue, the discriminant forms of $L_0(sr)$ and $L_0(\epsilon sr)$ must be isomorphic. By the oddity formula

$$\text{signature}(L) + \sum_{p \geq 3} p\text{-excess}(L) \equiv \text{oddity}(L) \pmod{8}.$$

In the proof of Lemma 8.11, we saw that the $p \neq 5$ parts of both discriminant forms are equal. So if we subtract the oddities of both forms we end up with

$$4 + 5\text{-excess}(L_0(sr)) - 5\text{-excess}(L_0(\epsilon sr)) \equiv 0 \pmod{8}.$$

In particular the discriminant forms cannot be isomorphic as their 5-excess differs.

Our next try, compatible with the oddity formula, is

$$L_0(sr) \oplus C \hookrightarrow U \oplus H_5 \oplus A_4 \oplus A_4.$$

and indeed this turns out to work.

Lemma 8.14. *There is a negative definite, root-free c_5 -lattice C with discriminant group isomorphic to $\mathcal{O}_K/r \times \mathcal{O}_K/(x-1)^3 \times \mathcal{O}_K/(x-1)$. Such that the determinant of the discriminant form on $\mathcal{O}_K/(x-1)^3$ is a non-square.*

Proof. We take C as a primitive extension of $L_0(sr) \oplus L_0(s)$ where the glue over the 5-part is isomorphic to $\mathcal{O}_K/(x-1)$. For this, we have to check that both sides have isomorphic forms on the 5 part. To do this we use the oddity formula. Recall that

$$p_i\text{-excess}(L_0(sr)) = 2(p_i - 1) + 4k_{p_i}$$

In the proof of Lemma 8.11 we have seen that $\det q_{p_i} = -1$. Write $p_i = 4k + r$, $0 \leq r < 4$. If $p_i \equiv 1 \pmod{4}$, the determinant -1 of q_{p_i} is a square mod p_i . Thus k_{p_i} vanishes and

$$p_i\text{-excess}(L_0(sr)) \equiv 2(4k + 0) \cdot 0 \equiv 0 \pmod{8}.$$

For $p_i \equiv 3 \pmod{4}$, we get

$$p_i\text{-excess}(L_0(sr)) \equiv 2(4k + 2) + 4 \cdot 1 \equiv 0 \pmod{8}.$$

Both lattices are negative definite and the $p \neq 5$ -excess and oddity vanish for both forms. From the oddity formula

$$0 \equiv 4 + 5\text{-excess} \equiv 4 + 3 \cdot (5 - 1) + 4k_5 \pmod{8}$$

we can see $k_5 = 0$ for both forms. We conclude that the determinant of each discriminant form over 5 is a square. This can be confirmed by a direct computation for $L_0(s)$. Now we may apply Lemma 8.13 to get the gluing and the condition on the determinants right.

It remains to check that C is root-free. First we remark that there are embeddings

$$L_0(sr) \hookrightarrow L_0(s) \hookrightarrow L_0 = A_4.$$

Suppose the sublattice $L_0(s)$ contains a root x . Then $(x, f(x), f^2(x), f^3(x))$ is a basis of $L_0(sr)$ consisting of roots, and hence $L_0(sr) = A_4$. This is impossible for determinant reasons. Secondly, notice that we glue over an isotropic subspace. This

implies that $C \hookrightarrow H_1 \oplus H_2$ for some even lattices H_i . Then any point of $h \in C$ which is not in $L_0(sr) \oplus L_0(s)$ can be written as $h = h_1 + h_2$ with $0 \neq h_i \in H_i$. In particular $h^2 = h_1^2 + h_2^2 \leq -2 - 2$. \square

Theorem 8.15. *There exists an infinite series of K3 surfaces admitting a symplectic and a non-symplectic automorphism of order 5. Their transcendental lattices are given as follows:*

let r_1, \dots, r_n be primes in \mathcal{O}_k over the distinct primes $p_1, \dots, p_n \equiv 1 \pmod{5}$. Let $s \in \mathcal{O}_k$ be the prime over 5. Then for $r = \prod_i r_i$,

$$T = L_0(sr).$$

Proof. We have to check the conditions of Proposition 8.10. (1) is Lemma 8.11. (2) We claim that there is a primitive extension of c_5 -lattices

$$L_0(sr) \oplus C \hookrightarrow U \oplus H_5 \oplus A_4 \oplus A_4 = T(\tau)$$

such that $T(\tau)^\vee/T(\tau) \cong (\mathcal{O}_K/(x-1))^3$. For this, take the C from the previous Lemma 8.14 which satisfies (3). The $p \neq 5$ part glues automatically by Theorem 3.7. It remains to check the 5-part of the construction. However, this is provided by Lemmas 8.13 and 8.14. \square

Remark 8.16. A similar construction with slightly different gluings also works for $p \equiv 4 \pmod{5}$.

9. GENERATORS OF THE NÉRON-SEVERI GROUP OF S

In general it is a hard problem to determine of the Néron-Severi lattice of a surface. More even so for explicit generators. As we have seen, automorphisms and elliptic fibrations come to help here.

9.1. Elliptic fibrations. A genus 1 fibration on a smooth surface X is a surjective morphism $f : X \rightarrow B$ to a smooth curve B such that the generic fiber is a smooth curve of genus one. It is known that for K3 surfaces the only possibility for the base is $B = \mathbb{P}^1$. If f admits a section $O : B \rightarrow X$, we call the pair (f, O) an elliptic fibration. This turns the generic fiber of an elliptic fibration E into an elliptic curve over the function field of the base curve B . Rational points of E correspond to sections of π and vice versa. We will not distinguish between the two concepts and call the resulting groups both the Mordell-Weil group of the elliptic fibration. It is denoted by MW.

The reason for us to consider elliptic fibrations is that their fibers give access to a good part of the Néron-Severi group. Together with the zero section O the fibers span the trivial lattice

$$\text{Triv}(X) := \langle O, \text{fiber components} \rangle_{\mathbb{Z}}.$$

It decomposes as an orthogonal direct sum of a hyperbolic plane spanned by O together with the fiber F and negative definite root lattices of type ADE consisting of fiber components (cf. [24, 50]). Note that the singular fibers (except in some cases in characteristics 2 and 3) are determined by the j -invariant and discriminant of the elliptic curve E .

The objects introduced so far are connected by the following theorem.

Theorem 9.1. [47] *There is a group isomorphism*

$$\mathrm{MW}(X) \cong \mathrm{NS}(X) / \mathrm{Triv}(X).$$

9.2. An elliptic fibration on S . Any $K3$ -surface with $\rho \geq 5$ admits a genus one fibration (cf. [42]). We have seen that the Picard number of S is $\rho = 18 \geq 5$. Thus S admits a genus one fibration. Since S is a $K3$ surface we know that the base curve is \mathbb{P}^1 .

The double cover \tilde{S} of \mathbb{P}^2 branched over $x_0(x_0^5 + x_1^5 + x_2^5)$ is given in affine charts by:

$$\begin{aligned} (x, z, y) &= \left(\frac{1}{s}, \frac{t}{s}, \frac{\tilde{y}}{s^k} \right) = \left(\frac{r}{q}, \frac{1}{q}, \frac{\hat{y}}{q^k} \right) \\ U : y^2 &= x^5 + z^5 + 1 & (x, y, z) &\mapsto (1 : x : z) \\ V : \tilde{y}^2 &= s(s^5 + t^5 + 1) & (s, \tilde{y}, t) &\mapsto (s : 1 : t) \\ W : \hat{y}^2 &= q(q^5 + r^5 + 1) & (q, \hat{y}, r) &\mapsto (q : r : 1) \end{aligned}$$

It has A_1 singularities over the points $(0 : 1 : -\zeta_5^k), k \in \{0, 1, 2, 3, 4\}$. Its minimal resolution S is given by a single blowup in each singular point.

Let $u = x + z$ then

$$y^2 - x^5 - z^5 - 1 = y^2 - (1 + u^5 - 5u^4x + 10u^3x^2 - 10u^2x^3 + 5ux^4)$$

is a curve of genus one over $\mathbb{C}(u)$ and in these coordinates a fibration is given by

$$\begin{aligned} \pi : S &\dashrightarrow \mathbb{P}^1 \\ (x, y, u) &\mapsto (1 : u). \end{aligned}$$

In the affine chart U one sees immediately 5 curves $C_k := (y + 1, x + \zeta_5^k z)$. We take C_1 as zero section.

Now that we have exhibited an elliptic fibration, let us compute its singular fibers. Using Cassels' formulas [13] and a computer algebra system one can compute the j -invariant and the discriminant.

$$\begin{aligned} j &= 2048 \frac{(u^5 + 6)^3}{(u^5 - 4)^2(u^5 + 16)} \\ \Delta &= -80(76\zeta_5^3 + 76\zeta_5^2 + 123) \frac{(u^5 + 16)(u^5 - 4)^2}{u^{21}} \end{aligned}$$

Together they determine the singular fibers and their type. The Euler numbers add up to 24 as befits a $K3$ surface.

$$5e(I_1) + 5e(I_2) + e(I_0^*) + e(III) = 5 \cdot 1 + 5 \cdot 2 + 6 + 3 = 24$$

Proposition 9.2. *The trivial lattice of (S, π) is isomorphic to*

$$L_\pi \cong U \oplus A_1^6 \oplus D_4.$$

It has rank 12 and determinant -2^8 .

Corollary 9.3. *(S, π) has Mordell-Weil rank 6.*

TABLE 6. Singular fibers of (S, π)

$k \in \{1, \dots, 5\}$	$(-\zeta_5^k \sqrt[5]{16} : 1)$	$(\zeta_5^k \sqrt[5]{4} : 1)$	$(0 : 1)$	$(1 : 0)$
$\nu(\Delta)$	1	2	6	3
behavior of j	$\nu(j) = -1$	$\nu(j) = -2$	$\nu(j) = 0$	$j(u = 0) = 1728$
Kodaira type	I_1	I_2	I_0^*	III

Proof. This is a direct consequence of $\rho = 18$ and the Shioda-Tate formula. \square

Let us fix some notation. We will give a prime divisor in terms of an ideal in an affine chart of \tilde{S} . The divisor is to be interpreted as the strict transform of the closure of this curve. Denote by E the exceptional divisor of the blowup in $s = t + 1 = 0$. It is a section. The elliptic fibration provides us with the following fiber components lying over the points $(1 : 0), (0 : 1)$ and $(\sqrt[5]{4} : 1)$:

$$C := (x + z, y + 1)$$

$$H := (s, \tilde{y})$$

$$D := (x + z - b, y + (2a^3 + 2a^2 + 1)5b^3z^2 + (a^3 + 6)b^4z + 9a^3 + 9a^2 + 10)$$

$$a := \zeta_5, b := \sqrt[5]{4}$$

Let us express two automorphisms of S in terms of their action on the affine open set U .

$$\tau : (x, y, z) \mapsto (x, y, \zeta_5 z)$$

$$\mu : (x, y, z) \mapsto (\zeta_5 x, y, \zeta_5 z)$$

Note that μ respects the fibration while τ does not.

Proposition 9.4. *The Néron-Severi group of the K3 surface defined by $y^2 = x^5 + z^5 + 1$ is integrally generated by the images of the curves C, D, E, H under the action of τ and μ .*

Proof. For example an integral basis is given by

$$\begin{aligned}
& \tau^k E & k \in \{0, 1, 2, 3, 4\}, \\
& H, \\
& \mu^k D & k \in \{0, 1, 2, 3, 4\}, \\
& \tau^k C & k \in \{1, 2, 3\}, \\
& \tau^k D & k \in \{1, 2\}, \\
& \tau \mu^k D & k \in \{1, 2\}.
\end{aligned}$$

One can compute their intersection matrix with a computer algebra system and obtain:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

As predicted it has signature $(1, 17)$ and determinant -125 . \square

Corollary 9.5. *The images of the trivial lattice L_π under μ, τ generate $\text{NS}(S)$ integrally.*

Corollary 9.6. *The Mordell-Weil group of $(S, \pi, \tau C)$ is of rank 6. It is generated by the sections*

$$\begin{aligned} \tau^k C & \quad k \in \{2, 3\}, \\ \tau^k D & \quad k \in \{1, 2\}, \\ \tau \mu^k D & \quad k \in \{1, 2\}, \\ E. \end{aligned}$$

where the two torsion is given by the relation $2E - 2\tau^2 C - 2\tau^3 C = 2(x + \zeta_5 z, y + 1) = 0$. This is the only relation.

Proof. By the theorem on the Mordell-Weil group, we take a basis for NS consisting of sections and fiber components. Then we discard the fibers and keep the sections. The D_4 fiber is H together with 4 exceptional divisors $\tau^k E$ $\tau \in \{1, 2, 3, 4\}$. The $\mu^k D$, $k \in \{0, 1, 2, 3, 4\}$ are part of the I_2 fibers. The rest are sections. The only fiber missing in the basis for NS is the type III fiber corresponding to $x + z = 0$. We take the component $G := (x + z, y - 1)$ not meeting the zero section and compute its intersection numbers with the chosen basis of NS. From this one may compute the basis representation of G and obtain $G \equiv 2E - 2\tau^2 C - 2\tau^3 C \pmod{L_\pi}$. This proves that $[\text{NS} : L_\pi] = 2$. A similar computation yields the two-torsion section $(x + \zeta_5 z, y + 1)$. \square

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